Coherent configurations as modules

Gejza Jenča, Anna Jenčová, Dominik Lachman

AAA 106, Olomouc, 2025

This research is supported by grants VEGA 2/0128/24 and 1/0036/23, APVV-20-0069.

The plan

1. Coherent configurations and association schemes.
2. String diagrams and Frobenius monoids.
3. Connections between them

Motivation: group actions on squares of sets

- Let $(\Gamma, ..e)$ be a group acting on a finite set X from the right: $\odot\colon X \times \Gamma \to X$ such that

$$(x \odot a) \odot b = x \odot (ab) \quad x \odot e = x.$$

Motivation: group actions on squares of sets

- Let $(\Gamma, ., e)$ be a group acting on a finite set X from the right: $\odot\colon X \times \Gamma \to X$ such that

$$(x \odot a) \odot b = x \odot (ab) \quad x \odot e = x.$$

- We may extend the action to $X \times X$ in an obvious way:

$$(x_1, x_2) \odot a := (x_1 \odot a, x_2 \odot a)$$

Motivation: group actions on squares of sets

- Let $(\Gamma, .., e)$ be a group acting on a finite set X from the right: $\odot \colon X \times \Gamma \to X$ such that

$$(x \odot a) \odot b = x \odot (ab) \quad x \odot e = x.$$

- We may extend the action to $X \times X$ in an obvious way:

$$(x_1, x_2) \odot a := (x_1 \odot a, x_2 \odot a)$$

- Clearly, this is an action of Γ, so it induces a decomposition of $X \times X$ into equivalence classes.

Motivation: group actions on squares of sets

- Let $(\Gamma, ..e)$ be a group acting on a finite set X from the right: $\odot\colon X \times \Gamma \to X$ such that

$$(x \odot a) \odot b = x \odot (ab) \quad x \odot e = x.$$

- We may extend the action to $X \times X$ in an obvious way:

$$(x_1, x_2) \odot a := (x_1 \odot a, x_2 \odot a)$$

- Clearly, this is an action of Γ, so it induces a decomposition of $X \times X$ into equivalence classes.
- Each of these equivalence classes is a subset of $X \times X$, that means, a relation on the set X.

Motivation: group actions on squares of sets

- Let $(\Gamma, ..e)$ be a group acting on a finite set X from the right: $\odot\colon X \times \Gamma \to X$ such that

$$(x \odot a) \odot b = x \odot (ab) \quad x \odot e = x.$$

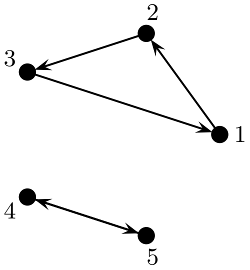
- We may extend the action to $X \times X$ in an obvious way:

$$(x_1, x_2) \odot a := (x_1 \odot a, x_2 \odot a)$$

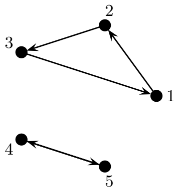
- Clearly, this is an action of Γ, so it induces a decomposition of $X \times X$ into equivalence classes.
- Each of these equivalence classes is a subset of $X \times X$, that means, a relation on the set X.
- Hence we obtain data in the form (X, S), where S is a system of relations.

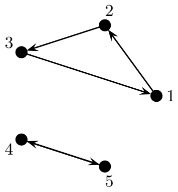
An example

Let $X = \{1, \ldots, 5\}$ and consider a permutation (an action of $\mathbb{Z}$) on X like this:

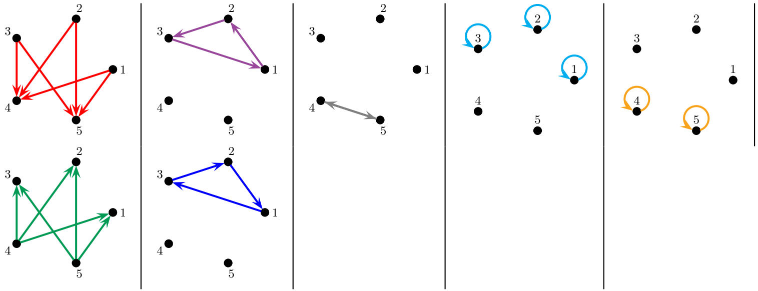


An example





The orbits look like this:



Coherent configurations

Definition

(Higman 1970) Let X be a finite set, let S be a system of subsets of $X \times X$ such that

(C1) S is a partition of $X \times X$.

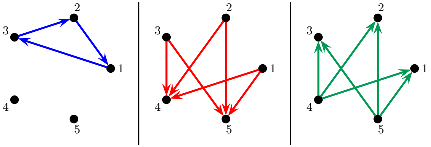
(C2) If $c \in S$ and $(x, x) \in c$, then c is a subset of the identity relation id_X. The elements $e \in S$ with $e \subseteq \mathrm{id}_X$ are called *units*. The set of all units is denoted by E_S.

(C3) For $a, b, c \in S$ and $(x, y) \in c$, the number ∇^c_{ab} of $z \in X$ such that $(x, z) \in a$ and $(z, y) \in b$ does not depend on the choice of $(x, y) \in c$.

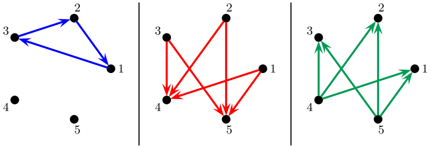
(C4) If $c \in S$, then $c^{-1} = \{(y, x) \colon (x, y) \in c\} \in S$

Then (X, S) is called *a coherent configuration on X*.

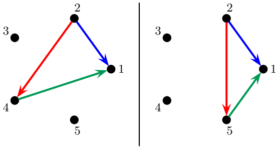
An illustration of (C3)



An illustration of (C3)



Over every • edge (x, y), there are exactly 2 •-• walks from x to y, for example $(x, y) = (1, 2)$:

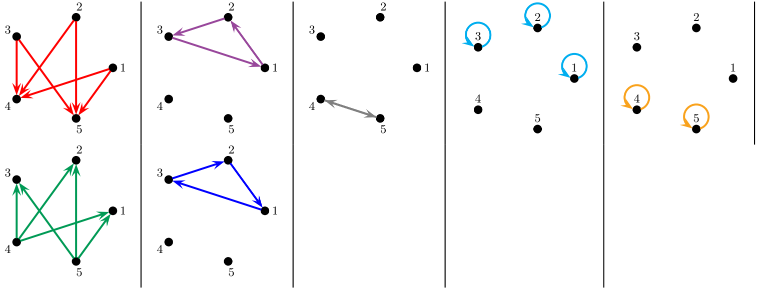


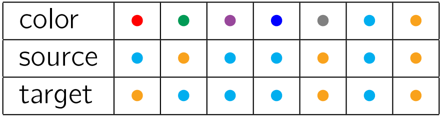
That means, $\nabla^{\bullet}_{\bullet\bullet} = 2$

Sources and targets of colors

- For every color $a \in S$, there is exactly one unit $p \in E_S$ such that $\nabla^p_{ab} > 0$.
- In this case, $b = a^{-1}$.
- This p is called the target of a and is denoted by $t(a)$.
- The source of a is defined analogously and denoted by $s(a)$.

Sources and target colors in our example





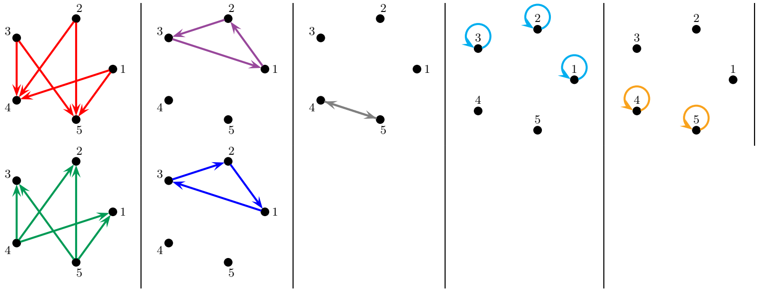
Valencies

- The valency of a color a is the number

$$\|a\| = \nabla^{s(a)}_{aa^{-1}}$$

- It is easy to see that whenever x is a vertex such that there is $(x, y) \in a$, $\|a\|$ is the number of a-colored edges sourced at a.

Valencies in our example



$$\|{\color{red}\bullet}\| = \nabla^{\color{cyan}\bullet}_{{\color{red}\bullet}{\color{red}\bullet}^{-1}} = \nabla^{\color{cyan}\bullet}_{{\color{red}\bullet}{\color{green}\bullet}} = 2$$

color	${\color{red}\bullet}$	${\color{green}\bullet}$	${\color{purple}\bullet}$	${\color{blue}\bullet}$	${\color{gray}\bullet}$	${\color{cyan}\bullet}$	${\color{orange}\bullet}$
valency	2	3	1	1	1	1	1

Association schemes

Definition

An *association scheme* is a coherent configuration (X, S) such that the identity relation

$$\mathrm{id}_X = \{(x, x) : x \in X\}$$

belong to S.

In other words, all the loops are of the same color.

Examples of association schemes

It is sometimes simpler to introduce in terms of of a coloring: take a surjective mapping

$$X \times X \to S.$$

and then identify the relations with the fibers of the mapping.

Example (Hamming scheme)

Let $X = 2^{\{1,2,\ldots,n\}}$, let $S = \{0, \ldots, n\}$ color the edge $(x, y) \in X \times X$ by the number of elements in the symmetric difference $(x \setminus y) \cup (y \setminus x)$.

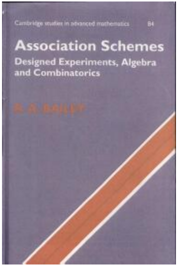
Example (Johnson scheme)

Restrict the Hamming scheme to k-element subsets.

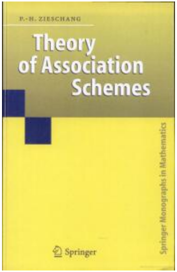
Example (Group scheme)

Let G be a group, let $X = G$, color the edge $(x, y) \in G \times G$ by the conjugacy class of xy^{-1}.

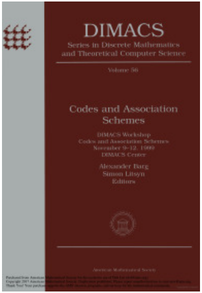
Books on association schemes



Bailey



Zieshang



Bark and Litsyn (eds.)

A book on coherent configurations

Chen,G. and Ponomarenko, I.:*Lectures on Coherent Configurations*, `https://pdmi.ras.ru/~inp/ccNOTES.pdf`

Monoidal categories

A monoidal category category is a category $\mathcal{C}$ equipped with

- a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- a chosen unit object I (behaving neutrally with respect to $\otimes$)
- natural isomorphisms α, λ, ρ,
- satisfying several conditions.

- $(\mathbf{Set}, \times, 1)$ - sets with direct product and a singleton set.
- $(\mathbf{Ab}, \otimes, \mathbb{Z})$ - abelian groups with tensor product and $\mathbb{Z}$.
- $(\mathbf{FinHilb}, \otimes, \mathbb{C})$ - finitely dimensional Hilbert spaces with tensor product and $\mathbb{C}$.
- $(\mathbf{Sup}, \otimes, \mathbf{2})$ - complete join semilattices, tensor product and the 2-chain.
- $(\mathbf{Rel}, \otimes, 1)$ - sets with direct product and a singleton set, but the morphisms are relations and not mappings.

String diagrams – motivation

- Suppose that we have a category $\mathcal{C}$, equipped with a monoidal structure $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$.
- This means that we have several complicated commutative diagrams (see MacLane) involving $\otimes$ and natural transformations, like

$$\begin{array}{ccccc}
((A\otimes B)\otimes C)\otimes D & \xrightarrow{\alpha_{A,B,C}\otimes \mathrm{id}_D} & (A\otimes(B\otimes C))\otimes D & \xrightarrow{\alpha_{A,B\otimes C,D}} & A\otimes((B\otimes C)\otimes D) \\
\downarrow{\scriptstyle \alpha_{A\otimes B,C,D}} & & & & \downarrow{\scriptstyle \mathrm{id}_A\otimes\alpha_{B,C,D}} \\
(A\otimes B)\otimes(C\otimes D) & & \xrightarrow[\alpha_{A,B,C\otimes D}]{} & & A\otimes(B\otimes(C\otimes D))
\end{array}$$

String diagrams – motivation

- The commutative diagrams are equations

String diagrams – motivation

- The commutative diagrams are equations, but

  $$\alpha_{A,B,C\otimes D}\circ\alpha_{A\otimes B,C,D} = \mathrm{id}_A\otimes\alpha_{B,C,D}\circ\alpha_{A,B\otimes C,D}\circ\alpha_{A,B,C}\otimes\mathrm{id}_D$$

  is even worse.
- We may choose to pretend that $(A\otimes B)\otimes C = A\otimes(B\otimes C)$ (strictification) but this does not solve everything:
- We still have do deal with trivial commutative diagrams like

$$\begin{array}{ccc} A\otimes B & \xrightarrow{f\otimes\mathrm{id}_B} & C\otimes B \\ {\scriptstyle \mathrm{id}_A\otimes g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C\otimes g} \\ A\otimes D & \xrightarrow[f\otimes\mathrm{id}_D]{} & C\otimes D \end{array}$$

String diagrams – motivation

String diagrams (in our context) are a way how to denote a morphism in a monoidal category by a picture so that

the trivial equalities become real equalities,

or, at the vary least, can be represented by

simple and intuitive deformations of one picture into another one.

String diagrams – motivation

The goal is rather mundane, but very useful:
we want to simplify computations.

String diagrams

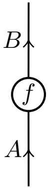
Objects, identities

A =  = id_A

String diagrams

Morphisms

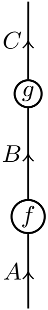
$$(f\colon A \to B) \;=$$



String diagrams

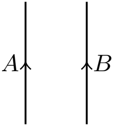
Composition of morphisms

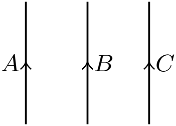
$$A \xrightarrow{f} B \xrightarrow{g} C \quad =$$



String diagrams

Tensor product of objects

$A \otimes B =$ 

$A \otimes (B \otimes C) = (A \otimes B) \otimes C =$ 

String diagrams

The identity object

$$I =$$

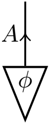
$$A \otimes I = I \otimes A =$$ 

$$= A$$

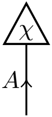
String diagrams

States and effects

$$I \xrightarrow{\phi} A \quad =$$



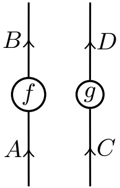
$$A \xrightarrow{\chi} I \quad =$$



String diagrams

Tensor product of morphisms

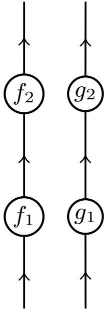
$$A \otimes C \xrightarrow{f \otimes g} B \otimes D \quad =$$



String diagrams

Tensor product is functorial

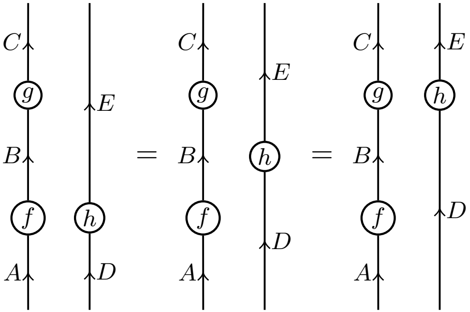
$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1)$$



String diagrams

Sliding

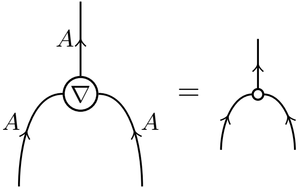
$$(g \otimes \mathrm{id}_E) \circ (f \otimes h) = (g \otimes \mathrm{id}_E) \circ (\mathrm{id}_B \otimes h) \circ (f \otimes \mathrm{id}_D) =$$
$$(g \otimes h) \circ (f \otimes \mathrm{id}_D)$$



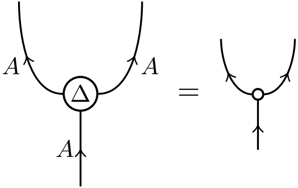
String diagrams

(Co)operations

$$A \otimes A \xrightarrow{\nabla} A \quad = $$



$$A \xrightarrow{\Delta} A \otimes A \quad = $$



Definition

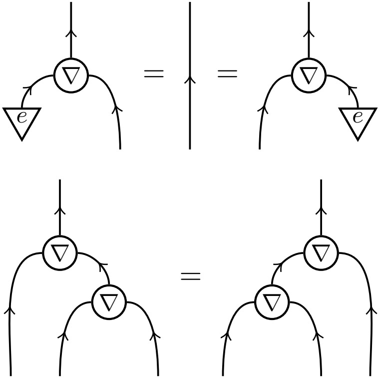
Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A *monoid* in $\mathcal{C}$ is a triple (S, ∇, e), where S is an object of $\mathcal{C}$, $e\colon I \to S$ and $\nabla : S \otimes S \to S$ such that the equations

$$\nabla \circ (e \otimes \mathrm{id}_S) = \lambda_S$$
$$\nabla \circ (\mathrm{id}_S \otimes e) = \rho_S$$
$$\nabla \circ (\nabla \otimes \mathrm{id}_S) = \nabla \circ (\mathrm{id}_S \otimes \nabla)$$

are satisfied.

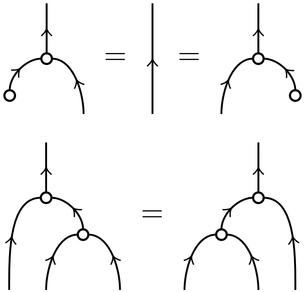
Monoids

With string diagrams



Monoids

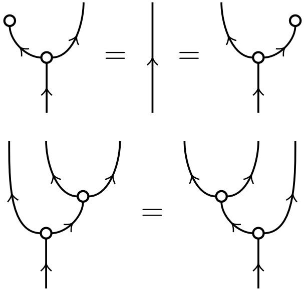
With string diagrams



Examples of monoids

- Monoids in $(\mathbf{Set}, \times, 1)$ are just the usual monoids.
- Monoids in $(\mathbf{Ab}, \otimes, \mathbb{Z})$ are rings.
- Monoids in $(\mathbf{Vect}_K, \otimes, K)$ are algebras.
- Monoids in $(\mathbf{Sup}, \otimes, \mathbf{2})$ are quantales.

Comonoids

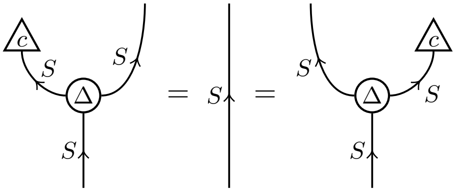


Comonoids in $(\mathbf{Set}, \times, 1)$?!

They exist, but are very rare.

Comonoids in $(\mathbf{Set}, \times, 1)$?!

They exist, but are very rare.



This already implies that

$$\Delta(a) = (a, a)$$

and it turns out that this Δ is coassociative.

Modules

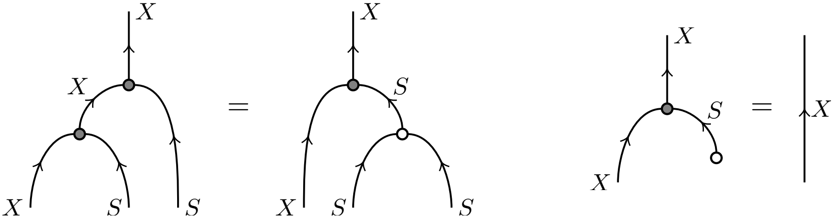
Let (S, ∇, e) be a monoid, let X be an object, let

$$X \otimes S \xrightarrow{\gamma} X$$

We say that γ is an S-module if

$$\gamma \circ (\gamma \otimes \mathrm{id}_S) = \gamma \circ (\mathrm{id}_X \otimes \nabla) \qquad \gamma \circ (\mathrm{id}_X \otimes e) = \lambda_X$$

are satisfied. In string diagrams, this is expressed by



where the gray dot represents the γ.

Frobenius monoids

A *Frobenius monoid* S is a structure

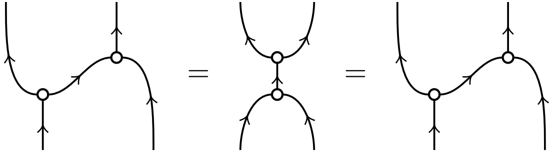
$$(S, \nabla, \Delta, e, c)$$

such that

- (S, ∇, e) is a monoid
- (S, Δ, c) is a comonoid
- $(\nabla \otimes \mathrm{id}_S) \circ (\mathrm{id}_S \otimes \Delta) = \Delta \circ \nabla = (\mathrm{id}_S \otimes \nabla) \circ (\Delta \otimes \mathrm{id}_S)$

Frobenius monoids

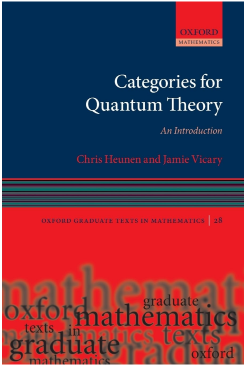
$$(\nabla \otimes \mathrm{id}_S) \circ (\mathrm{id}_S \otimes \Delta) = \Delta \circ \nabla = (\mathrm{id}_S \otimes \nabla) \circ (\Delta \otimes \mathrm{id}_S)$$



Many examples of Frobenius monoids

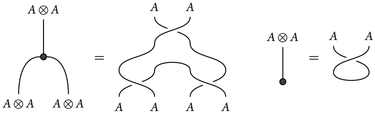
- Every cohomology ring is a Frobenius monoid in the category of abelian groups with $\otimes$.
- Every finite dimensional C^*-algebra is a Frobenius monoid in the category of finitely dimensional Hilbert spaces with $\otimes$.
- Every small groupoid is a Frobenius monoid in the category of sets and relations with $\times$.
- Every Lambek pregroup is a Frobenius monoid in the category of posets and monotone relations with $\times$.
- Every effect algebra is a Frobenius monoid in the same category.
- Every Frobenius monoid in $(\mathbf{Set}, \times, 1)$ is trivial (singleton).

The string diagrams book we use

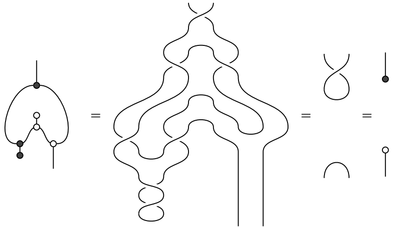


Heunen, C. and Vicary, J., 2019. *Categories for Quantum Theory: an introduction.* Oxford University Press.

A page from that book



Then a straightforward graphical calculation shows the following:



The other identity in (6.4) follows similarly.

The category of nonnegative matrices

The category $\mathbf{Mat}_{\mathbb{R}_0^+}$ is the category whose

- objects are finite sets and
- morphisms from a set X to a set Y are matrices of elements of $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \leq 0\}$ with columns indexed by the set X and rows indexed by the set Y.
- For $f : X \to Y$, we denote the element of the matrix corresponding to a pair $x \in X$, $y \in Y$ by f_x^y.
- The composition of morphisms is the usual matrix multiplication: for $f : A \to B$ and $g : B \to C$, we have

$$(g \circ f)_a^c = \sum_{b \in B} f_a^b g_b^c.$$

- and the identity morphism $\mathrm{id}_A \colon A \to A$ is the unit matrix.

This is not linear algebra

Proposition

In $\mathbf{Mat}_{\mathbb{R}_0^+}$, $f\colon A \to B$ is an isomorphism iff for every $a \in A$ there is a unique $b \in B$ such that $f_a^b > 0$ and for every $b \in B$ there is a unique $a \in A$ such that $f_a^b > 0$.

In other words, if we want to construct all isomorphisms in $\mathbf{Mat}_{\mathbb{R}_0^+}$, we may simply take all permutation matrices and replace 1 in the matrix by some arbitrary positive real number.

Tensor products

There is a symmetric monoidal structure $(\mathbf{Mat}_{\mathbb{R}_0^+}, \otimes, I)$:

- I is a one-element set (we write $I := \{*\}$),
- for finite sets X, Y, $X \otimes Y := X \times Y$,
- for morphisms $f : A \to B$ and $g : C \to D$, $f_1 \otimes f_2 : A \otimes C \to B \otimes D$ is the tensor (or Kronecker) product of matrices:

$$(f \otimes g)^{bd}_{ac} = f^b_a . g^d_c$$

$\mathbf{Mat}_{\mathbb{R}_0^+}$ is a dagger compact category

There is additional structure on $\mathbf{Mat}_{\mathbb{R}_0^+}$:

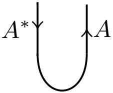
- For every morphism $f\colon A \to B$ there is a morphism $f^\dagger\colon B \to A$ (the transpose of the matrix f).
- For every object A there is
  - a dual object A^* (we may take $A = A^*$),
  - a morphism $\eta_A\colon I \to A^* \otimes A$,
  - a morphism $\varepsilon_A\colon A \otimes A^* \to I$,

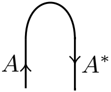
  subject to several conditions.
- $\mathbf{Mat}_{\mathbb{R}_0^+}$ is a *dagger compact category*.
- Other examples: **FinHilb**, **Rel**, **RelPosInv**, ...

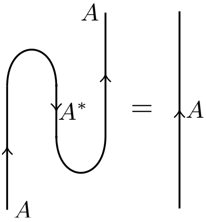
String diagrams in dagger compact categories

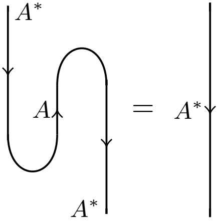
- If we have a dagger compact category,
- we are allowed to do additional transformations on string diagrams:
    - bending strings
    - straightening strings
    - turning things upside down.

Snake equations

$\eta_A =$ 

$\varepsilon_A =$ 





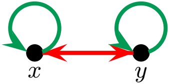
Back to coherent configurations

Let X be a finite set, let S be a set of relations on X that is a partition of $X \times X$. We may encode the data into a $0,1$ matrix γ such that

- columns are indexed by $X \otimes S$
- rows are indexed by X
- the values are given by the rule

$$\gamma_{xa}^{y} = \begin{cases} 1 & \text{if } (x, y) \in a \\ 0 & \text{otherwise} \end{cases}$$

An example



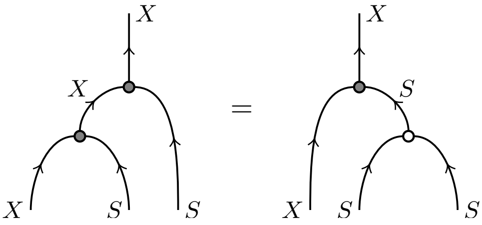
γ	($\color{red}{\bullet}$,x)	($\color{red}{\bullet}$,y)	($\color{green}{\bullet}$,x)	($\color{green}{\bullet}$,y)
x	0	1	1	0
y	1	0	0	1

A natural question

- The matrix γ is a morphism in $\mathbf{Mat}_{\mathbb{R}_0^+}$

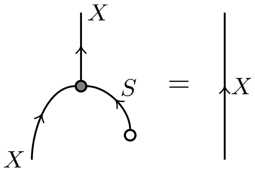
$$X \otimes S \xrightarrow{\gamma} X$$

- It has a "datatype" of an S-module over X.
- For which (X, S) is there a monoid (S, ∇, e) such that $\gamma : X \otimes S \to X$ is an S-module?



means that

(C3) For $a, b, c \in S$ and $(x, y) \in c$, the number ∇^c_{ab} of $z \in X$ such that $(x, z) \in a$ and $(z, y) \in b$ does not depend on the choice of $(x, y) \in c$.



means that

(C2) If $c \in S$ and $(x, x) \in c$, then c is a subset of the identity relation id_X.

Main result part 1

But these are exactly the axioms of a coherent configuration!

Theorem

Let X be a finite set, let S be a partition of $X \times X$, so that (X, S) satisfies (C1). Let $\gamma \colon X \otimes S \to X$ be the $\mathbf{Mat}_{\mathbb{R}_0^+}$-morphism associated with (X, S). Then γ is a module over some pointed magma (S, ∇, e) if and only if (X, S) satisfies (C2) and (C3) of the definition of a coherent configuration.

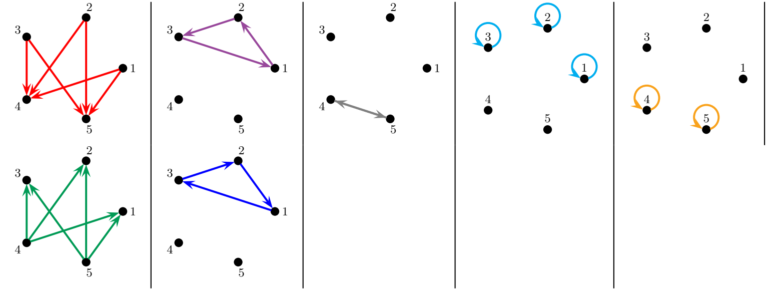
Main result part 1

But these are exactly the axioms of a coherent configuration!

Theorem

Let X be a finite set, let S be a partition of $X \times X$, so that (X, S) satisfies (C1). Let $\gamma\colon X \otimes S \to X$ be the $\mathbf{Mat}_{\mathbb{R}_0^+}$-morphism associated with (X, S). Then γ is a module over some pointed magma (S, ∇, e) if and only if (X, S) satisfies (C2) and (C3) of the definition of a coherent configuration.Moreover, (S, ∇, e) is then a monoid.

Example



One column in the matrix ∇:

∇	●●
●	0
●	0
●	2
●	2
●	0
●	2
●	0

A snippet of the proof

The associative law in $\mathbf{Mat}_{\mathbb{R}_0^+}$ for a binary operation ∇ on S means that, for all $a, b, c, d \in S$,

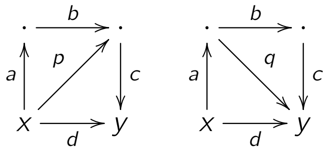
$$\sum_{p \in S} \nabla^d_{pc} \nabla^p_{ab} = \sum_{q \in S} \nabla^d_{aq} \nabla^q_{bc}.$$

A snippet of the proof

The associative law in $\mathbf{Mat}_{\mathbb{R}_0^+}$ for a binary operation ∇ on S means that, for all $a, b, c, d \in S$,

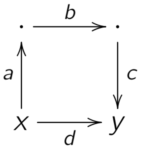
$$\sum_{p \in S} \nabla^d_{pc} \nabla^p_{ab} = \sum_{q \in S} \nabla^d_{aq} \nabla^q_{bc}.$$

For a coherent configuration, this has a combinatorial meaning



A snippet of the proof

So two sides of the associative equality just count the number of the squares



in two different ways.

Main result part 2

What about the

(C4) If $c \in S$, then $c^{-1} = \{(y, x) \colon (x, y) \in c\} \in S$?

Main result part 2

What about the

(C4) If $c \in S$, then $c^{-1} = \{(y, x) : (x, y) \in c\} \in S$?

It turns out that it exactly encodes the fact that (S, ∇, e) is a monoid part of a Frobenius monoid with $c = e^\dagger$:

Theorem

Let (X, S) be a pair of finite sets satisfying (CC1)-(CC3). Then $e^\dagger$ is a counit of some Frobenius monoid $(S, \nabla, e, \Delta, e^\dagger)$ if and only if (X, S) is a coherent configuration.

Coassociativity of Δ means something

The comultiplication $\Delta\colon S \to S \otimes S$ is given by

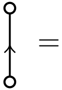
$$\Delta_a^{bc} = \frac{1}{\|b^{-1}\|} \nabla^c_{b^{-1}a}$$

The fact that this is coassociative has a combinatorial interpretation.

Some additional axioms

There are subclasses of Frobenius monoids. What are their corresponding subclasses of coherent configurations?

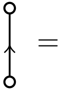
- The "extra" axiom is $e \circ e^\dagger = \mathrm{id}_I$.



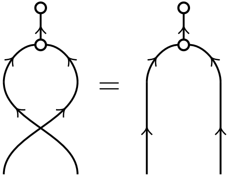
Some additional axioms

There are subclasses of Frobenius monoids. What are their corresponding subclasses of coherent configurations?

- The "extra" axiom is $e \circ e^{\dagger} = \mathrm{id}_I$.



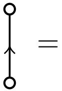
- These are exactly the association schemes.
- The "symmetric" axiom is



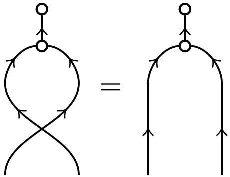
Some additional axioms

There are subclasses of Frobenius monoids. What are their corresponding subclasses of coherent configurations?

- The "extra" axiom is $e \circ e^{\dagger} = \mathrm{id}_I$.



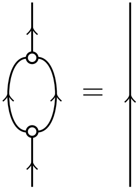
- These are exactly the association schemes.
- The "symmetric" axiom is



- This means that $\|a\| = \|a^{-1}\|$, for every $a \in S$.

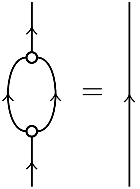
Some additional axioms

- The "special" axiom is $\nabla \circ \Delta = \mathrm{id}_S$



Some additional axioms

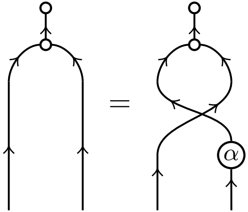
- The "special" axiom is $\nabla \circ \Delta = \mathrm{id}_S$



- This means that $\|a\| = 1$ for every $a \in S$ (thin coherent configurations).

Nakayama automorphism

- Every Frobenius monoid S in a compact category has a special automorphism uniquely determined by the property



- In our case, $\alpha\colon S \to S$ is a diagonal matrix

$$\alpha_a^b = \begin{cases} \frac{\|a\|}{\|a^{-1}\|} & a = b \\ 0 & \text{otherwise} \end{cases}$$

- The fact that α is an automorphism means that

$$\frac{\|c\|}{\|a\|.\|b\|} = \frac{\|c^{-1}\|}{\|a^{-1}\|.\|b^{-1}\|}$$

whenever $\nabla_{ab}^c > 0$.

Dagger Frobenius monoids

- These are Frobenius monoids satisfying

$$c = e^{\dagger} \quad \Delta = \nabla^{\dagger}$$

- In our case, we have $c = e^{\dagger}$, but not the other property.
- Can we "daggerize" our Frobenius monoids?
- We will see after the break...

Thank you for your attention.