Regular maps of given hyperbolic type with no non-trivial exponents

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Abstract

Existence of orientably-regular but chiral maps of arbitrary hyperbolic type is known as a consequence of a general theorem of Jones [Chiral covers of hypermaps, *Ars Math. Contemp.* 8 (2015), 425–431] and more specific theorems by Conder, Hucíková, Nedela and Širáň [Chiral maps of given hyperbolic type, *Bull. London Math. Soc.* 48 (2016), 38–52], with proofs relying respectively on holomorphic differentials and permutation groups defined by coset diagrams. An extension to existence of orientably-regular maps of any hyperbolic type with no exponent except 1 was obtained recently by Bachratá and Bachratý [Orientably regular maps of given hyperbolic type with no non-trivial exponents, *Ann. Comb.* 27 (2023) 353–372] with the help of canonical covers of maps. Using parallel products of maps, we give a short proof of the latter extension. Combining this approach with constructions of maps on linear fractional groups, we also establish sufficient conditions for existence of *non-orientable* regular maps of an arbitrary hyperbolic type with no exponent except ± 1.

1 Introduction

Foundations of the theory of orientable maps, that is, cellular embeddings of graphs on orientable surfaces, were laid decades ago in [13]. The part of the theory dealing with maps exhibiting the 'highest level of symmetry' together with a survey of later developments can be found in [20]. We begin by outlining the basics of the theory of orientably and fully regular maps, freely quoting facts from the resources [13, 20].

A map M on a compact orientable surface is *orientably-regular* if the group $\mathrm{Aut}^+(M)$ of all its orientation-preserving automorphisms is a regular permutation group on the set of arcs (edges with direction) of M. The concept of orientable regularity formalises the intuitive notion of a map with the 'highest level of orientation-preserving symmetry'. Regularity implies that both the carrier surface as well as the underlying graph of M are connected; for simplicity we will assume that the latter contains no semi-edges. Another consequence of regularity is that all vertices of M have the same valency, say k, and all face boundary walks of M have the same length, say m, in which case we use the traditional Schläfli symbol $\{m, k\}$ to denote the *type* of M.

Let ε be an edge of M incident with a vertex v, and suppose the carrier surface $\mathcal{S}$ of M is oriented anti-clockwise. By orientable regularity, there is an orientation-preserving automorphism x of M that takes an arc arising from ε onto its reverse. This automorphism is uniquely determined, has order 2 and acts on $\mathcal{S}$ like a 180-degree rotation about the centre of ε. Similarly, there is a unique element $y \in \mathrm{Aut}^+(M)$ of order k acting on $\mathcal{S}$ as a k-fold rotation about v that takes ε to the anti-clockwise next edge on $\mathcal{S}$ incident with v. The composition xy then represents an m-fold rotation of M about the centre of a face incident with ε. By connectedness, the group $\mathrm{Aut}^+(M)$ is *generated* by the two elements x and y, and hence it admits a presentation of the form $\mathrm{Aut}^+(M) = \langle\, x, y \mid x^2,\ y^k,\ (xy)^m, \dots \rangle$. It follows that $\mathrm{Aut}^+(M)$ is a smooth quotient of the ordinary $(2, k, m)$-triangle group $\Delta(2, k, m)$, with presentation $\langle\, X, Y \mid X^2,\ Y^k,\ (XY)^m \,\rangle$. Equivalently, $\mathrm{Aut}^+(M) \cong \Delta(2, k, m)/K$ for some normal, torsion-free subgroup K of $\Delta(2, k, m)$. This subgroup K is the *map subgroup* described in [13]. Conversely, any such (finite) smooth quotient of $\Delta(2, k, m)$ determines an orientably-regular map of type $\{m, k\}$.

By regularity of the action of $\mathrm{Aut}^+(M) = \langle x, y\rangle$ on the set of arcs (incident vertex-edge pairs) of M, the map itself may be identified with the group $G = \mathrm{Aut}^+(M)$ in such a way that arcs, edges, vertices and faces of M are identified respectively with elements of $G = \langle x, y\rangle$ and right cosets of the subgroups $\langle x\rangle$, $\langle y\rangle$ and $\langle xy\rangle$, with incidence given by non-empty intersection, and the action of G given by right multiplication. It is therefore standard in the theory of maps to identify an orientably-regular map M of type $\{m, k\}$ with the *presentation* of the group G of orientation-preserving automorphisms of M in the form $G = \langle\, x, y \mid x^2,\ y^k,\ (xy)^m, \dots \rangle$, and in such a case we will simply write $M = (G; x, y)$, with $G = \mathrm{Aut}^+(M)$.

It follows that questions about orientably-regular maps can be reduced to purely group-theoretic questions. This also applies to 'external symmetries', which are not not induced by a map automorphism. Perhaps the simplest example of such a situation is self-duality. If $M = (G; x, y)$ is an orientably-regular map, its dual is the (orientably-regular) map $M^* = (G; x, xy)$, and the existence of a map isomorphism of M onto M^* reduces to the existence of an automorphism of the *group* G fixing x and interchanging y with xy. A similar example can be given with self-Petrie duality, but here we will focus on a different kind of external symmetry that we now describe.

Let $k \geq 2$ and e be integers, with e relatively prime to k. An orientably-regular map $M = (G; x, y)$ of valency k is said to have a 'hole symmetry' of exponent e if there is a group automorphism of G fixing x and taking y to y^e. For an insight into this concept in the orientable case (with an anti-clockwise oriented supporting surface), suppose that one re-embeds the underlying graph of an orientably-regular map $M = (G; x, y)$ of valency k by replacing, for every vertex v, the local anti-clockwise cyclic permutation π_v of arcs emanating from v by its power π_v^e for a fixed e coprime to k. The new map, which may be called the e-th *rotational power* of M and denoted M^e, might or might not be isomorphic to the original map M, but if it is, we call e an *exponent* of M, arriving at the 'hole symmetry' introduced above. The collection of all exponents of M mod k forms a group under multiplication (a subgroup of the group of units mod k) called the *exponent group* of M; see [15].

An important special case arises when $e = -1$. For an orientably-regular map $M = (G; x, y)$, having exponent $e = -1$ means existence of an automorphism of G inverting both x and y (and indeed fixing the involution x) . This is equivalent to the map $M = (G; x, y)$ being isomorphic to its *mirror image* $M^{-1} = (G; x, y^{-1})$, and then M is described as being *fully regular*. Orientable maps that are not isomorphic to their mirror image are said to be *chiral*, which is equivalent to -1 *not* being an exponent of the map.

Orientably-regular maps with 'large' external symmetry groups appear to be extremely rare. Infinite families of such maps admitting all possible exponents for a given valency ('kaleidoscopic symmetry') and all self-dualities ('trinity symmetry') were obtained in [2] for every even valency. Constructions of k-valent orientably-regular maps with a given subgroup of units mod k as their exponent group can be found in [7]. By a remark made in [21], however, this cannot be extended to maps of *arbitrary* hyperbolic type $\{m, k\}$, that is, for any given m and k such that $1/k + 1/m < 1/2$.

Here we are concerned with the extreme situation at the other end of the spectrum, by considering orientably-regular maps of given hyperbolic type $\{m, k\}$, but with no non-trivial hole symmetries. Such maps are necessarily chiral, but the requirement of triviality of the exponent group is much stronger. Restriction to hyperbolic types is natural since there are no chiral orientably-regular maps on a sphere, and the requirement of triviality of the exponent group for toroidal orientably-regular maps reduces to their chirality.

Existence of orientably-regular maps of any given hyperbolic type with no exponent except 1 was recently established by Bachratá and Bachratý in [3], with the help of covers of maps with no non-trivial exponents but no other symmetry requirements. An essentially equivalent approach but with the help of an adaptation of the coset diagrams of [8] was used in [1]. For completeness, and to connect triviality of the exponent groups to chirality, existence of infinitely many orientably-regular but chiral maps of any given hyperbolic type was established in [8] and independently in [12].

In this paper we give a very short proof of the existence of orientably-regular maps of given hyperbolic type but with trivial exponent group, using parallel products of maps, as introduced in [22]. Since a non-orientable regular map automatically admits the exponent -1, the best one can hope for are non-orientable regular maps of given hyperbolic type but with no exponent except ± 1. Our main results are sufficient conditions for existence of such maps, obtained by combining parallel products with exploration of regular maps on fractional linear groups over finite fields (and in particular, those of characteristic 2).

The paper is organised as follows. In Section 2 we present our main ingredients, which are regular maps on linear fractional groups over finite fields, together with basic facts on parallel products of regular maps and the associated automorphism groups. A short proof of the existence of orientably-regular maps with any given hyperbolic type but with no non-trivial exponents is then given in Section 3. In Section 4 we develop machinery for recognition of exponents distinct from ± 1 in non-orientable regular maps over linear fractional groups over finite fields of characteristic 2. Based on that, in Section 5 we prove our main theorems containing sufficient conditions for existence of non-orientable regular maps of any hyperbolic type but with no exponents distinct from ± 1.

2 Ingredients

As indicated, we will use some of the known theorems about orientably-regular and fully regular maps with automorphism group isomorphic to linear fractional groups over finite fields, that is, $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$ for some prime-power q. We will represent elements of $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$ in the usual manner, that is, by non-singular 2×2 matrices, with determinant 1 and equivalent up to sign in the first case, and equivalent up to multiplication by any non-zero element of the field $\mathrm{GF}(q)$ in the second case. To ease the notation, the symbol $\mathrm{diag}(\alpha,\beta)$ will stand for an element in either of these two groups, with α,β in the main diagonal and zeros as off-diagonal elements. Let $\{m,k\}$ be a hyperbolic pair and let p be a prime dividing neither k nor m, and for now let us assume that $p\neq 2$. By elementary number theory, there exists a power q' of p such that the field $\mathrm{GF}(q')$ contains primitive $2k$-th and $2m$-th roots of unity; let ξ and η be such roots and let $D(\xi,\eta)=\xi^2+\xi^{-2}+\eta^2+\eta^{-2}$. By [17] and a more detailed elaboration in [9], the following hold:

> (*) *Letting $y=\pm\mathrm{diag}(\xi,\xi^{-1})\in\mathrm{PSL}(2,q')$, the same group contains an involution x such that the product xy has trace $\pm(\eta+\eta^{-1})$. Moreover, let $q=p^s$ be the smallest power of p such that the field $\mathrm{GF}(q)$ contains both $\xi+\xi^{-1}$ and $\eta+\eta^{-1}$. If $D(\xi,\eta)\neq 0$, which is equivalent to $-\xi^2\neq\eta^2,\ \eta^{-2}$, the group $\langle\, x,y\mid x^2,y^k,(xy)^m,\dots\rangle$ is conjugate either to $\mathrm{PSL}(2,q)$ or, under certain conditions and with s even, to $\mathrm{PGL}(2,p^{s/2})=\mathrm{PGL}(2,\sqrt{q})$.*

Another important observation we need here was made in [18, Theorem 3], implying that for any x and y as above there exists an involutory element c fixing x and inverting y by conjugation; moreover, if $\langle x,y\rangle\cong\mathrm{PGL}(2,\sqrt{q})$, then $c\in\langle x,y\rangle$. A consequence of this is that if $H=\langle x,y\rangle$ with $c\in H$, then H is the automorphism group of a non-orientable regular map M of type $\{m,k\}$. We will denote the non-orientable regular map arising this way by $\mathrm{Map}(\xi,\eta)$.

In Section 4 we will review and develop more particular details about the maps $\mathrm{Map}(\xi,\eta)$ in the special case when q is a power of 2. In the meantime we introduce two more facts on which our constructions will be based. The first one is a modified form of Lemma 6.1 and Corollary 6.2 of [11], and we accompany it with a short proof.

Lemma 1. *Let K and L be normal subgroups of a group G. If $G = KL$, then $G/(K \cap L) \cong G/K \times G/L$. In particular, this holds when K and L do not contain each other and G/K or G/L is simple.*

Proof. First, let $J = K \cap L$, which is normal in G, and set $\overline{H} = H/J$ for every subgroup H of G containing J. Then $\overline{G} = \overline{KL} = \overline{K}\,\overline{L}$, with $\overline{K} \cap \overline{L} = \overline{J}$ being trivial, and therefore $\overline{G} \cong \overline{K} \times \overline{L}$. Also by the Second Isomorphism Theorem, $\overline{K} = K/(K \cap L) \cong KL/L = G/L$ and $\overline{L} = L/(K \cap L) \cong KL/K = G/K$, so $G/(K \cap L) \cong G/L \times G/K$.

In particular, if G/K or G/L is simple, then K or L is a maximal normal subgroup of G, and hence if K and L do not contain each other, then $G = KL$ and the conclusion holds. □

Let $M = (G; x, y)$ and $N = (H; r, s)$ be a pair of orientably-regular maps. The *parallel product* $G \,||\, H$ of the two groups is the subgroup of $G \times H$ generated by the ordered pairs (x, r) and (y, s), and the *parallel product* $M \,||\, N$ of the two maps is defined by setting $M \,||\, N = (G \,||\, H; (x, r), (y, s))$. Parallel products of maps were introduced in [22], and they correspond to map subgroups as follows. For simplicity, suppose that both M and N are of the same type $\{m, k\}$, and let $\Delta = \Delta(2, k, m)$ be the ordinary triangle group from the previous section. Let K and L be the map subgroups corresponding to M and N, that is, $\mathrm{Aut}^+(M) \cong \Delta/K$ and $\mathrm{Aut}^+(N) \cong \Delta/L$. Then $\mathrm{Aut}^+(M \,||\, N) \cong \Delta/(K \cap L)$, which is the basic property of the parallel products of maps presented in [22]. It follows that for such maps, M is a cover of N (which is equivalent to $K < L$) if and only if $M \,||\, N = M$, a fact also observed in [22]. The concept of a parallel product of maps extends to fully regular maps in a natural way, and we will omit the details.

In subsequent sections we will make use of the following consequence of Lemma 1 for parallel products of regular maps.

Proposition 1. *Let M and N be non-isomorphic maps with the same hyperbolic type $\{m, k\}$, both orientably-regular, or both fully regular and non-orientable, but not a cover of each other, and let $M \,||\, N$ be their parallel product. If M, N are orientably-regular and at least one of the groups* $\mathrm{Aut}^+(M)$ *and* $\mathrm{Aut}^+(N)$ *is simple, then* $\mathrm{Aut}^+(M \,||\, N) \cong \mathrm{Aut}^+(M) \times \mathrm{Aut}^+(N)$. *Similarly, if M are N are fully regular and non-orientable, and at least one of* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *is simple, then* $\mathrm{Aut}(M \,||\, N) \cong \mathrm{Aut}(M) \times \mathrm{Aut}(N)$.

Proof. Let $\Delta = \Delta(2, k, m)$ be the triangle group with presentation $\langle\, X, Y \mid X^2, Y^k, (XY)^m \,\rangle$, and let K and L be the map subgroups of Δ for the maps M and N respectively, so that $\mathrm{Aut}^+(M) \cong \Delta/K$ and $\mathrm{Aut}^+(N) \cong \Delta/L$. By the above description of parallel products, we have $\mathrm{Aut}^+(M \,||\, N) \cong \Delta/(K \cap L)$, with K and L not containing each other because M and N are not a cover of each other. Then since at least one of Δ/K and Δ/L is assumed to be simple, Lemma 1 implies that

$$\mathrm{Aut}^+(M \,||\, N) \cong \Delta/(K \cap L) \cong \Delta/K \times \Delta/L \cong \mathrm{Aut}^+(M) \times \mathrm{Aut}^+(N)\ .$$

The argument for fully regular non-orientable maps M and N is analogous, with Δ taken to be the extended $(2,k,m)$-triangle group $\langle\, A,B,C \mid A^2, B^2, C^2, (AC)^2,\ (BC)^k, (CA)^m \,\rangle$. $\square$

The second fact we will use is a combination of a restricted version of Corollary 3.12 in [4] and a special case of Theorem 3.1 in [5], on automorphism groups of direct products.

Proposition 2. *If two finite groups G and H, each with trivial centre, have no common non-trivial direct factor, then* $\mathrm{Aut}(G\times H)\cong \mathrm{Aut}(G)\times\mathrm{Aut}(H)$. *If G is a finite simple group, then* $\mathrm{Aut}(G\times G)\cong(\mathrm{Aut}(G)\times\mathrm{Aut}(G))\rtimes C_2$, *where the C_2 factor interchanges the the two copies of* $\mathrm{Aut}(G)$.

3 Orientably regular maps of arbitrary hyperbolic type with no non-trivial exponent

We begin by proving a construction of fully regular orientable maps with any given hyperbolic type, having no exponents except ±1. Existence of such maps was proved in [19] with the help of residual finiteness of triangle groups. Here we offer an explicit constructive proof of this fact.

Theorem 1. *For any given hyperbolic pair $\{m,k\}$, there exist infinitely many primes p and infinitely many finite reflexible orientably-regular maps M of type $\{m,k\}$ with* $\mathrm{Aut}^+(M)$ *isomorphic to* $\mathrm{PSL}(2,p)$, *such that the only exponents of M are ±1.*

Proof. Let $p=p(k,m)$ be an odd prime congruent to 1 modulo each of k and m. Existence of an infinite number of such primes is a consequence of Dirichlet's theorem, which gives infinitely many primes congruent to 1 mod km. By [14] or [17], there exist elements $x,y\in\mathrm{PSL}(2,p)$ such that x has trace 0, while y and xy respectively have traces $\pm(\xi+\xi^{-1})$ and $\pm(\eta+\eta^{-1})$ for some $2k$-th and $2m$-th primitive roots ξ and η mod p, making y and xy have orders k and m, with $G=\langle x,y\rangle=\mathrm{PSL}(2,p)$. Let $M=(G;x,y)=\mathrm{Map}(\xi,\eta)$ be the corresponding orientably-regular map of type $\{m,k\}$.

The automorphism group of $\mathrm{PSL}(2,p)$ is known to be isomorphic to $\mathrm{PGL}(2,p)$, which contains $\mathrm{PSL}(2,p)$ as a (normal) subgroup of index 2. Hence if $M=\mathrm{Map}(\xi,\eta)$ admitted an exponent e relatively prime to k and such that $2\le e\le k-2$, then there would be an element $u\in\mathrm{PGL}(2,p)$ conjugating y to y^e. By [9], the elements y and y^e may be identified with matrices $\pm\mathrm{diag}(\xi,\ \xi^{-1})$ and $\pm\mathrm{diag}(\xi^e,\xi^{-e})$ in $\mathrm{PSL}(2,p^2)$. But it is well known that two such matrices are conjugate if and only if they have the same trace (up to sign), and a simple calculation (as in the proof of Lemma 4.4 of [9]) shows that this happens if and only if $e=\pm1$ mod k. It follows that every such map $M=\mathrm{Map}(\xi,\eta)$ has no non-trivial exponents. $\square$

We are now ready to prove our main theorem on orientably-regular maps.

Theorem 2. *For any given hyperbolic pair (k,m), there exists an orientably-regular map of type $\{m,k\}$ with no exponent except* 1.

Proof. Let M be a map of type $\{m, k\}$ as constructed in the proof of Theorem 1, with $\mathrm{Aut}^+(M)$ isomorphic to the simple group $\mathrm{PSL}(2, p)$ for some sufficiently large p. Also let N be an orientably-regular but chiral map of the same type $\{m, k\}$, as constructed in [8], with $\mathrm{Aut}^+(N)$ isomorphic to the symmetric group S_n or the alternating group A_n, for some $n > 6$. Clearly, for sufficiently large p, the two maps are not a cover of each other. Now consider the parallel product $M \,||\, N$ of these two maps. By Proposition 1, we know that $\mathrm{Aut}^+(M \,||\, N) \cong \mathrm{Aut}^+(M) \times \mathrm{Aut}^+(N)$. This means that if $G = \mathrm{Aut}^+(M) = \langle x, y \mid x^2, y^k, (xy)^m, \dots \rangle$ and $H = \mathrm{Aut}^+(N) = \langle r, s \mid r^2, s^k, (rs)^m, \dots \rangle$, then the group $\mathrm{Aut}^+(M \,||\, N) \cong G \times H$ is generated by the elements (x, r) and (y, s), of orders 2 and k, with product (xy, rs) of order m.

Next, suppose that some unit $e \bmod k$ is an exponent of $M \,||\, N$. This implies existence of an automorphism of $G \times H$ fixing (x, r) and mapping (y, s) to $(y, s)^e = (y^e, s^e)$. Then since G is a simple group distinct from H, Proposition 2 tells us that such an automorphism is formed by a pair of automorphisms of G and H, taking $(x, y) \mapsto (x, y^e)$ and $(r, s) \mapsto (r, s^e)$ respectively. This implies that e is an exponent of both M and N. By Theorem 1, however, we find that $e \in \{1, -1\}$, while on the other hand, chirality of N implies that $e \neq -1$. We conclude that $e = 1$, and hence that the orientably-regular map $M \,||\, N$ of type $\{m, k\}$ has no non-trivial exponent. □

As the reader may have noticed, in the proof of Theorem 2 the map M was chosen in order to restrict exponents to 1 and -1, while the map N was chosen to eliminate -1 as a possibility. This shows how the parallel product construction can be very helpful in constructing orientably-regular maps with a particular exponent group.

4 Non-orientable regular maps on linear fractional groups over fields of characteristic two

In characteristic two, for any $n \geq 1$ the group $\mathrm{PSL}(2, 2^n) \cong \mathrm{PGL}(2, 2^n) \cong \mathrm{SL}(2, 2^n)$ has the striking property that its only elements of even order are involutions. As it turns out (just as in [17, 9]), the construction of maps $M(\xi, \eta)$ given in the highlighted summary (*) in Section 2 carries over to characteristic 2, except that in that case the elements ξ and η are k-th and m-th primitive roots in some (possibly larger) field of characteristic 2, with q again being the smallest power of 2 containing both $\xi+\xi^{-1}$ and $\eta+\eta^{-1}$.

To recall a few details which will be needed later, let $\{m, k\}$ be a hyperbolic type with both entries odd, and let ξ and η be primitive k-th and m-th roots of unity in some field $\mathrm{GF}(q')$ of characteristic 2, such that $\xi \neq \eta, \eta^{-1}$, and then define $D = \xi+\xi^{-1}+\eta+\eta^{-1}$. Furthermore, let $q = 2^n$ be the smallest power of 2 containing both $\xi + \xi^{-1}$ and $\eta + \eta^{-1}$. This condition is known to be equivalent to n being the smallest positive integer such that k divides one of $2^n - 1$ and $2^n + 1$, and also m divides one of $2^n - 1$ and $2^n + 1$. In particular, k and m divide $2^n \pm 1$.

Explicit calculations undertaken in [9] imply that the following choice of elements $x, y \in \mathrm{SL}(2, q')$ determines a group with presentation $\langle\, x, y \mid x^2, y^k, (xy)^m, \dots \rangle$, conjugate to $\mathrm{SL}(2, q)$ and isomorphic to the automorphism group of a regular map $\mathrm{Map}(\xi, \eta)$:

$$x = x(\xi, \eta) = \frac{1}{\xi+\xi^{-1}} \begin{pmatrix} \eta+\eta^{-1} & D^2\xi \\ \xi^{-1} & \eta+\eta^{-1} \end{pmatrix} \ , \quad y = y(\xi, \eta) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \ . \tag{1}$$

(Note here that $\eta+\eta^{-1} \neq 0$ because a finite field of characteristic 2 has no element of multiplicative order 2, and $D \neq 0$ because $\xi \neq \eta, \eta^{-1}$.) Moreover, the map $\mathrm{Map}(\xi, \eta)$ is non-orientable, since the element c from Section 2 conjugating x and y to their inverses must belong to $\mathrm{SL}(2, q)$, see [14, 9].

We will now investigate possible exponents of the map $M = \mathrm{Map}(\xi, \eta)$ of 'odd' type $\{m, k\}$ introduced above, with $\mathrm{Aut}(M) \cong \langle x, y\rangle \cong \mathrm{SL}(2, q)$ from (1). This requires a closer look at automorphisms of the group $\langle x, y\rangle$, which we do next.

For $q = 2^n$, the automorphism group of $\mathrm{SL}(2, q)$ is known to be a split extension of $\mathrm{SL}(2, q)$ by the cyclic group of Galois automorphisms of $\mathrm{GF}(q)$. If k divides $q-1 = 2^n-1$, then the group $G = \langle x, y\rangle$ for x, y given by (1) coincides with the unique subgroup J of $\mathrm{SL}(2, q^2)$ isomorphic to $\mathrm{SL}(2, q)$, which consists of unimodular matrices *defined over* $\mathrm{GF}(q)$, and then all automorphisms of G come from the split extension described above. But if k divides $2^n + 1$, then $G = \langle x, y\rangle$ is a conjugate of J contained in $\mathrm{SL}(2, q')$ but distinct from J. In this situation, let $h \in \mathrm{SL}(2, q')$ be such that $hGh^{-1} = J$ and let $J \mapsto \alpha J^\theta \alpha^{-1}$ be an automorphism of J for some $\alpha \in \mathrm{SL}(2, q)$ and a Galois automorphism θ induced by $z \mapsto z^{2^i}$ for some $i \in \{0, 1, \dots, n-1\}$ and all $z \in \mathrm{GF}(q)$. Note that θ induces also an automorphism of $\mathrm{SL}(2, q')$ when applied to elements $z \in \mathrm{GF}(q') > \mathrm{GF}(q)$. Now substituting $hGh^{-1} = J$ in $J \cong \alpha J^\theta \alpha^{-1}$ gives $hGh^{-1} \cong \alpha(hGh^{-1})^\theta \alpha^{-1}$, which implies that $G \cong \beta G^\theta \beta^{-1}$ for $\beta = h^{-1}\alpha h^\theta$. So, even in the case where k divides $2^n + 1$, all automorphisms of $G = \langle x, y\rangle$ have the form $u \mapsto \beta u^\theta \beta^{-1}$ for suitable $\beta \in \mathrm{SL}(2, q')$ and some Galois automorphism θ of the field $\mathrm{GF}(q)$.

Hence in both cases $e \neq \pm 1$ is an exponent of $\mathrm{Map}(\xi, \eta)$ if and only if there is an element $\alpha \in \mathrm{SL}(2, q')$ and a Galois automorphism $\theta : \ z \mapsto z^{2^i}$ of $\mathrm{GF}(q)$ for some $i \in \{1, \dots, n-1\}$ such that

$$\alpha x^\theta = x\alpha \quad \text{and} \quad \alpha y^\theta = y^e \alpha \ , \tag{2}$$

where x and y are given by (1). (We may exclude $i = 0$ from consideration because then α would conjugate y to y^e, and we would find $e = \pm 1$ by the trace argument used before.)

The second equation of (2) gives two kinds of solutions, one of the form $\alpha = \mathrm{diag}(d^{-1}, d)$ for some non-zero $d \in GF(q')$ and some Galois automorphism given by $\theta(z) = z^{2^i}$ such that $2^i \equiv e \bmod k$, and the other in the form of an off-diagonal matrix and with Galois automorphism $\theta(z) = z^{-2^i}$ with $e \equiv -2^i \bmod k$. We may disregard the latter, since it arises as a composition of the former with the (unique) automorphism of $\mathrm{SL}(2, 2^n)$ inverting both x and y (induced by conjugation by the off-diagonal matrix with entries $D\xi$ and $(D\xi)^{-1}$). Note also that e is an exponent of $\mathrm{Map}(\xi, \eta)$ if and only if $-e$ is. Continuing to apply

$\alpha = \mathrm{diag}(d^{-1}, d)$ and $\theta(z) = z^{2^i}$ and comparing entries in the products appearing in the first equation in (2) one finds that $\alpha x^\theta = x\alpha$ is equivalent to the following three equations:

$$\left(\frac{\eta+\eta^{-1}}{\xi+\xi^{-1}}\right)^{2^i} = \frac{\eta+\eta^{-1}}{\xi+\xi^{-1}}\,,\quad \left(\frac{D^2\xi}{\xi+\xi^{-1}}\right)^{2^i} = \frac{d^2D^2\xi}{\xi+\xi^{-1}}\,,\quad \text{and}\quad \left(\frac{\xi^{-1}}{\xi+\xi^{-1}}\right)^{2^i} = \frac{d^{-2}\xi^{-1}}{\xi+\xi^{-1}}\,. \tag{3}$$

A further calculation (details of which we omit) reveals that the first equation of (3) implies equivalence of the second and third equations of (3), and taking into account the facts that $2^i \equiv e \bmod k$ and every non-zero element of $\mathrm{GF}(q')$ has a unique square root, the third equation finally gives $d = (\xi^e{+}1)/(\xi{+}1)$.

Of importance here is the first equation of (3), which states that $\rho^{2^i} = \rho$ for the ratio $\rho = \rho(\xi,\eta) = (\eta{+}\eta^{-1})/(\xi{+}\xi^{-1})$. Note also that $\rho \neq 1$ because of the condition $\xi \neq \eta,\ \eta^{-1}$. The order $o(\rho)$ of ρ is then a divisor of $2^i{-}1$, but obviously $o(\rho)$ is also a divisor of $2^n{-}1$, and hence a divisor of $\gcd(2^i{-}1, 2^n{-}1) = 2^j{-}1$, where $j = \gcd(i,n)$. Note here that j divides n by the well known fact that $2^j{-}1$ divides $2^n{-}1$ if and only if j divides n, and our assumption on the range of i implies that $j \neq n$. But from this point on, working backwards and letting $e \geq 1$ be a positive integer smaller than k such that $e \equiv 2^j \bmod k$ for j as above, we find that e is also an exponent of $\mathrm{Map}(\xi,\eta)$. As $\rho^{2^j} = \rho$, the ratio $\rho(\xi,\eta)$ is an element of the proper subfield $\mathrm{GF}(2^j)$ of $\mathrm{GF}(2^n)$.

It follows that the order of the *smallest* subfield containing $\rho(\xi,\eta)$ is an exponent of $\mathrm{Map}(\xi,\eta)$, and, conversely, the smallest power of 2 which (mod k) is an exponent of $\mathrm{Map}(\xi,\eta)$ is the order of a subfield containing $\rho(\xi,\eta)$; this smallest power is then a generator of a (cyclic) subgroup of the exponent group of $\mathrm{Map}(\xi,\eta)$, namely the subgroup induced by involving the Galois automorphism as above. More precisely, let 2^ℓ be the order of the smallest subfield containing $\rho(\xi,\eta)$. Since the order of ξ divides $2^n \pm 1$, there is a smallest positive integer t such that $2^{t\ell} \equiv 1 \bmod k$, and then $t\ell$ is a divisor of n if k divides $2^n{-}1$, and a divisor of $2n$ if k divides $2^n{+}1$. Then if e satisfies $2^\ell \equiv e \bmod k$ and $1 \leq e \leq k-1$, the units $e, e^2, \ldots, e^{t-1}$ form a cyclic group of exponents of $\mathrm{Map}(\xi,\eta)$, of order t, induced by automorphisms of $\langle x,y\rangle$ defined with the help of the Galois action; and moreover, there are no other exponents of $\mathrm{Map}(\xi,\eta)$ of this kind.

It remains to clarify the role of $e = -1 \equiv k-1 \bmod k$ which, as we know, is always an exponent of $\mathrm{Map}(\xi,\eta)$ arising from conjugation inverting x and y. Can the same exponent be obtained also by an automorphism of $\langle x,y\rangle$ of the form $u \mapsto \alpha u^\theta \alpha^{-1}$ for some $\alpha \in \langle x,y\rangle$ and some non-trivial Galois automorphism θ of the field $\mathrm{GF}(2^n)$, as considered above?

By the first equation of (3) and its consequences, θ would need to have the form $z \mapsto z^{2^i}$ for some i such that $2^i \equiv -1 \bmod k$. Note, however, that if $2^i \equiv -1 = e \bmod k$, so that k divides $2^i + 1$, then the first equation of (3) reduces to $(\eta + \eta^{-1})^{2^i} = \eta + \eta^{-1}$, which is equivalent to $(\eta^{2^i} + \eta)(1 + \eta^{-(2^i+1)}) = 0$ and hence to m dividing one of $2^i \pm 1$. But we have assumed at the very beginning that the smallest positive i such that each of k and m divides $2^i \pm 1$ is n. For $i = n$, however, the Galois automorphism $z \mapsto z^{2^n}$ appearing in (3) is trivial (giving $d = 1$). It follows that $e = -1$ never arises as an exponent from involving Galois conjugation $z \mapsto z^{2^i}$ for $i \in \{1, 2, \ldots, n-1\}$.

Collecting the above arguments yields a proof of the following statement.

Proposition 3. *Let $\{m, k\}$ be a hyperbolic type with odd entries, let ξ and η be primitive k-th and m-th roots in some field of characteristic 2 such that $\xi \neq \eta$, η^{-1}, and let $\mathrm{Map}(\xi, \eta)$ be the corresponding non-orientable regular map, with $q = 2^n$ being the smallest power of 2 such that $\mathrm{GF}(2^n)$ contains both $\xi+\xi^{-1}$ and $\eta+\eta^{-1}$. Also let ℓ be a positive divisor of n such that 2^ℓ is the smallest order of a subfield of $\mathrm{GF}(2^n)$ containing the ratio $\rho(\xi, \eta) = (\eta+\eta^{-1})/(\xi+\xi^{-1})$, and let t be the smallest positive integer such that $2^{t\ell} \equiv 1 \bmod k$. Then every exponent of $\mathrm{Map}(\xi, \eta)$ has the form $2^{j\ell}$ or $-2^{j\ell} \bmod k$ for some $j \in \{0, 1, \dots, t-1\}$, and the exponent group of $\mathrm{Map}(\xi, \eta)$ is isomorphic to the direct product $C_t \times C_2$.* □

With the above notation, we have the following obvious consequence of Proposition 3.

Corollary 1. *A non-orientable regular map $M = \mathrm{Map}(\xi, \eta)$ with $\mathrm{Aut}(M) \cong \mathrm{SL}(2, 2^n)$ has no exponents distinct from ± 1 if and only if the ratio $\rho(\xi, \eta)$ is contained in no proper subfield of $\mathrm{GF}(2^n)$.* □

5 Non-orientable regular maps of hyperbolic type with almost trivial exponent group

We begin by proving four consequences of Proposition 3 and Corollary 1 as a preparation towards constructions of non-orientable regular maps with no exponents except ± 1 with help of parallel products of maps.

Proposition 4. *For every odd integer $k \geq 5$, there exists a non-orientable regular map of type $\{k, k\}$ having no exponent distinct from ± 1.*

Proof. Let n be the smallest positive integer such that k divides $2^n \pm 1$, let ξ be a corresponding primitive k-th root of 1 in $\mathrm{GF}(2^n)$ or $\mathrm{GF}(2^{2n})$, and let $\eta = \xi^2$. Note that $\xi \neq \eta$, η^{-1} since $k \geq 5$. From $\eta + \eta^{-1} = (\xi + \xi^{-1})^2$ it follows that $\mathrm{GF}(2^n)$ is the smallest field of characteristic 2 containing $\xi + \xi^{-1}$ (and of course also $\eta + \eta^{-1}$). As the ratio $\rho(\xi, \eta)$ is now simply equal to $\xi + \xi^{-1}$, the non-orientable regular map $\mathrm{Map}(\xi, \eta)$ constructed in Proposition 3 has no non-trivial exponents, by Corollary 1. □

Proposition 5. *For every odd integer $k \geq 7$, there exists a non-orientable regular map of type $\{3, k\}$ such that neither the map nor its dual has no exponents distinct from ± 1.*

Proof. Let η be a primitive 3rd root of unity in some finite field of characteristic 2. Then η is a root of the equation $\eta^2 + \eta + 1 = 0$, and so $\eta + \eta^{-1} = 1$. Next let n be the smallest positive integer such that k divides $2^n \pm 1$. Then n is also the smallest positive integer for which $\mathrm{GF}(2^n)$ contains $\xi + \xi^{-1}$ (and $1 = \eta + \eta^{-1}$) for every choice ξ of a primitive k-th root of unity. Now if 2^ℓ were a non-trivial exponent of $M(\xi, \eta)$ induced by the Galois action, with the smallest positive ℓ such that $1 \leq \ell < n$, then by (3) and Proposition 3, the smallest subfield containing $\xi + \xi^{-1}$ would be $\mathrm{GF}(2^\ell)$. But by minimality of n this would imply that $\ell = n$, a contradiction. Thus $M(\xi, \eta)$ has no non-trivial exponents, as does its dual (because it has valency 3). □

Proposition 6. *For every odd integer $k \geq 7$, there exists a non-orientable regular map of type $\{5, k\}$ such that neither the map nor its dual has an exponent distinct from ± 1.*

Proof. Let η be a primitive 5th root of unity in some finite field of characteristic 2, and let $\nu = \eta + \eta^{-1}$. Then since η satisfies the equation $\eta^4 + \eta^3 + \eta^2 + \eta + 1 = 0$, it follows that $\nu^2 + \nu + 1 = 0$, so that ν has multiplicative order 3, and $\nu^4 = \nu$. Next let ξ be a primitive k-th root of unity, again in some finite field of characteristic 2, and also let n be the the smallest integer greater than 1 such that both 5 and k divide $2^n \pm 1$. Observing that 5 divides $2^j \pm 1$ if and only if j is even or $j = 1$, we see that n must be even.

Now suppose that $M(\xi, \eta)$ has a non-trivial exponent. Then by Proposition 3 we may assume that this exponent has the form 2^ℓ for some proper divisor ℓ of n. If ℓ is even, then from $\nu^4 = \nu$ it follows that $\nu^{2^\ell} = \nu$, and then the leftmost part of (3) implies that $\xi + \xi^{-1}$ is contained in $\mathrm{GF}(2^\ell)$, but then $\ell = n$ by the properties of n, a contradiction. Hence ℓ must be odd, and then the leftmost part of (3) reduces to

$$(\xi + \xi^{-1})^{2^\ell - 1} = \nu \ . \tag{4}$$

Moreover, since the order of ν (namely 3) does not divide $2^\ell - 1$ for odd ℓ, it follows from (4) that the order of $\xi + \xi^{-1}$ is a divisor of $2^{2\ell} - 1$. But now 2ℓ cannot be a proper divisor of n, as this would contradict minimality of n with respect to 5 and k dividing $2^n \pm 1$. We conclude that for η and ξ as above, if the map $M(\xi, \eta)$ has a non-trivial exponent, then it has the form 2^ℓ for $\ell = n/2$, with $n/2$ odd.

Let us now mimic the above considerations for our chosen η but with ξ replaced by ξ^2, assuming that 2^ℓ is a non-trivial exponent of the map $M(\xi^2, \eta)$ for some proper divisor ℓ of n. (Note that the minimal n here is the *same* as above.) It can be checked that (4) will then have the form

$$(\xi^2 + \xi^{-2})^{2^\ell - 1} = \nu \tag{5}$$

and the conclusion that $\ell = n/2$ remains the same for the map $M(\xi^2, \eta)$. But then comparison of (4) and (5) gives $(\xi + \xi^{-1})^{2^\ell - 1} = 1$, which contradicts (4). It follows that one of the maps $M\xi, \eta)$ and $M(\xi^2, \eta)$ has only trivial exponents, proving the existence of non-orientable regular maps of hyperbolic type $\{5, k\}$ with only trivial exponents.

For the dual, observe that if a map of hyperbolic type $\{k, 5\}$ with automorphism group isomorphic to $\mathrm{SL}(2, 2^n)$ for some n admitted a non-trivial exponent e, then e would be 2 or 3 ($\equiv -2$) mod 5, and the exponent group would be cyclic of order 4, which contradicts the final assertion of Proposition 3. □

Proposition 7. *For every hyperbolic pair (k, m) with distinct odd entries $k, m \geq 7$ and such that $\gcd(k, m) \in \{3, 5\}$ there exists a non-orientable regular map of type $\{m, k\}$ with no exponent distinct from ± 1.*

Proof. Let k and m be as in the statement, with $\gcd(k, m) = d \in \{3, 5\}$; we will assume that one of the two values of d is fixed in what follows. Propositions 5 and 6 then guarantee existence of non-orientable regular maps $M_1 = (G_1; x_1, y_1)$ of type $\{d, k\}$ and

$M_2 = (G_2; x_2, y_2)$ of type $\{m, d\}$ for suitable groups $G_i \cong \mathrm{SL}(2, 2^{n_i})$ for $i \in \{1, 2\}$, both having ± 1 as the only exponents. Our assumptions also imply that the two maps do not cover each other but note that the two groups may be abstractly isomorphic, e.g. for $k = 39$ and $m = 105$, with $n_1 = n_2 = 12$ and $d = 3$. For the parallel product $M = M_1 \,||\, M_2$ our earlier Proposition 1 implies that $\mathrm{Aut}(M) \cong G_1 \times G_2$ and so $M = (G; x, y)$ with $G \cong G_1 \times G_2$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Further, by our assumptions, for $i \in \{1, 2\}$ the order of y, that is, the valency of M, is equal to $\mathrm{lcm}(k, d) = k$ and the order of $xy = (x_1y_1, x_2y_2)$, the face length of M, is equal to $\mathrm{lcm}(d, m) = m$. It follows that the resulting (non-orientable) map M is of type $\{m, k\}$.

Now, if e is an exponent of M, then the assignment $x \mapsto x$ and $y \mapsto y^e$ extends to an automorphism of G. If G_1 and G_2 are not isomorphic, Proposition 2 tells us that $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2)$. In this case, for $i \in \{1, 2\}$ the above assignment would give rise to automorphisms of G_i fixing x_i and sending y_i to y_i^e. In other words, e would be an exponent of both M_1 and M_2 and hence $e = \pm 1$. If $G_1 \simeq G_2$, Proposition 2 implies that the only other possibility for the assignment $x \mapsto x$ and $y \mapsto y^e$ to extend to an automorphism of G is to exchange y_1 with y_2^e and y_2 with y_1^e, which is impossible by orders of these elements. □

We are now ready to address existence of non-orientable regular maps of a given hyperbolic type $\{m, k\}$ having no exponent other than ± 1. We begin with the case when at least one of k and m is even, using a direct construction.

Theorem 3. *Let (k, m) be a hyperbolic pair with at least one even entry. Then, there exists a non-orientable regular map of type $\{m, k\}$ with no exponent distinct from ± 1.*

Proof. Suppose that at least one of k, m is even. Then by Theorem 2 of [16] and its proof, there is an infinite set of odd primes p congruent to 1 mod both $2k$ and $2m$, such that for any $2k$-th and $2m$-th primitive roots ξ and η mod p the map $M = \mathrm{Map}(\xi, \eta)$ is regular and non-orientable, and has type $\{m, k\}$, and $\mathrm{Aut}(M) \cong \mathrm{PGL}(2, p) \cong \langle x, y \rangle$, where $y = \pm\mathrm{diag}(\xi, \xi^{-1})$. The fact that the only exponents of M are ± 1 follows almost verbatim from the second part of the proof of our Theorem 1 in Section 3. □

If both entries of a hyperbolic pair (k, m) are odd, we can only offer a partial result for k and m that are not relatively prime. This will be done by a combination of Propositions 1 and 2 from Section 2, supported by the material developed in Section 4 together with Propositions 4 and 7, extending the tricks used in the corresponding proofs.

Theorem 4. *Let (k, m) be a hyperbolic pair with odd and non-coprime entries. Then, there exists a non-orientable regular map of type $\{m, k\}$ with ± 1 as the only exponent.*

Proof. Let both entries k and m in our hyperbolic pair be odd and let $d = \gcd(k, m) > 1$. The situations when $k = m \geq 5$ or $d = 3$ are treated in Propositions 4 and 7 and so we will assume that for the odd highest common factor d is such that $d \geq 5$. Also, Proposition 6 allows us to assume that both $k, m \geq 7$ in what follows.

Let ξ and η be primitive k-th and m-th roots of unity in a finite field of characteristic 2, and let $q = 2^n$ be the smallest power of 2 such that the field $F = \mathrm{GF}(q)$ contains both $\xi + \xi^{-1}$ and $\eta + \eta^{-1}$. Since $(z + z^{-1})^2 = z^2 + z^{-2}$ in $\mathrm{GF}(q)$ and this field is closed under taking square roots (which are unique), it follows that the same field F is also the smallest containing both $\xi + \xi^{-1}$ and $\eta^2 + \eta^{-2}$. Our assumption that $k \neq m$ implies that both η and η^2 are distinct from ξ and ξ^{-1}.

It follows that $M = \mathrm{Map}(\xi, \eta)$ and $N = \mathrm{Map}(\xi, \eta^2)$ are non-orientable regular maps, with the same type $\{m, k\}$, and with both having automorphism group isomorphic to $G = \mathrm{SL}(2, q)$. We next show that M and N are not isomorphic (and hence not a cover of each other, albeit having the same automorphism group). To demonstrate this, observe that a map isomorphism from M to N would have to be induced by an automorphism of $G = \mathrm{SL}(2, q)$, that is, by composition of conjugation by some element of G with a Galois automorphism of G, with the consequence that the pairs of traces $(\xi + \xi^{-1}, \eta + \eta^{-1})$ and $(\xi + \xi^{-1}, \eta^2 + \eta^{-2})$ corresponding to M and N would have to be related by the same Galois automorphism. We show that the latter implies $d = 3$, contrary to our assumption.

Suppose that a Galois automorphism of F, of the form $z \mapsto z^{2^\ell}$ for some $\ell \in \{1, ..., n-1\}$ and every $z \in F$, fixes $\xi + \xi^{-1}$ but sends $\eta + \eta^{-1}$ to $\eta^2 + \eta^{-2}$. Observe first that, in general, for any given non-zero element $u \in F$ of multiplicative order $\mathrm{ord}(u)$ and every integers i, j, one has $u^i + u^{-i} = u^j + u^{-j}$ if and only if $(u^{i+j} + 1)(u^{i-j} + 1) = 0$, which is equivalent to $\mathrm{ord}(u)$ dividing one of $i + j$, $i - j$, commonly written in the form $\mathrm{ord}(u) \mid i \pm j$. Using this, with $\mathrm{ord}(\xi) = k$ and $\mathrm{ord}(\eta) = m$ the condition $\xi^i + \xi^{-i} = \xi + \xi^{-1}$ for $i = 2^\ell$ and $j = 1$ translates to $k \mid 2^\ell \pm 1$, and the condition $\eta^i + \eta^{-i} = \eta^2 + \eta^{-2}$ for the same $i = 2^\ell$ with $j = 2$ similarly translates to $m \mid 2^\ell \pm 2$. But $d = \gcd(k, m)$ shares the divisibility properties of both k and m, that is, $d \mid 2^\ell \pm 1$ and $d \mid 2^\ell \pm 2$, which implies that $d \in \{1, 3\}$, a contradiction. It follows that M is not isomorphic to N if $d \geq 5$.

Our next aim is to show that M and N cannot have the same exponent $e \neq \pm 1$. So suppose the contrary, and let $e \neq \pm 1$ be a common exponent of both M and N. By Proposition 3 we may assume that $e = 2^i$, where i is the smallest *proper* divisor of n such that both $\rho(\xi, \eta)$ and $\rho(\xi, \eta^2)$ are contained in $\mathrm{GF}(2^i)$. Then $\rho(\xi, \eta^2)/\rho(\xi, \eta) = \eta + \eta^{-1}$ is an element of $\mathrm{GF}(2^i)$, as is $(\eta + \eta^{-1})/\rho(\xi, \eta) = \xi + \xi^{-1}$. But this means that both $\xi + \xi^{-1}$ and $\eta + \eta^{-1}$ are contained in a proper subfield of $\mathrm{GF}(2^n)$, which is a contradiction to the minimality of $q = 2^n$. It follows that the only common exponents of M and N are ± 1.

Next, let $M = (G; x, y)$ be a representation of $M = \mathrm{Map}(\xi, \eta)$ in the form $G = \langle x, y \mid x^2, y^k, (xy)^m, \ldots \rangle$, with x and y given by (1). For the map $N = \mathrm{Map}(\xi, \eta^2)$ we may use the same automorphism y, but with a modification x' of x obtained by replacing η with η^2 in (1). Then we may represent N in the form $N = (G; x', y)$ for the same group G but with presentation $G = \langle x', y \mid (x')^2, y^k, (x'y)^m, \ldots \rangle$. The maps M and N are distinct and their automorphism groups are both isomorphic to the simple group $G \cong \mathrm{SL}(2, q)$, and so it follows from Proposition 1 that the automorphism group of the parallel product $M \,||\, N$ is isomorphic to $G \times G$. Furthermore, by Proposition 2, the automorphism group of $\mathrm{Aut}(M \,||\, N)$ is isomorphic to $(\mathrm{Aut}(M) \times \mathrm{Aut}(N)) \rtimes C_2$, with the C_2-part inducing a transposition of the two factors.

Now suppose that this parallel product has an exponent $e \neq \pm 1$. Then just as in the proof of Theorem 2, there exists an automorphism γ of the group $\mathrm{Aut}(M \,||\, N)$ fixing the pair (x, x') and sending the pair (y, y) to $(y, y)^e = (y^e, y^e)$. By the final observation in the previous paragraph, γ is induced either by isomorphisms $M \to M^e$ and $N \to N^e$ such that $(x, y) \mapsto (x, y^e)$ and $(x', y) \mapsto (x', y^e)$, or by isomorphisms $M \to N^e$ and $N \to M^e$ such that $(x, y) \mapsto (x', y^e)$ and $(x', y) \mapsto (x, y^e)$.

In the first case M and N would have the same exponent $e \neq \pm 1$, a possibility that has been excluded. In the second case, composing the two isomorphisms $M \to N^e$ and $N \to M^e$ in both ways, that is, $M \to N^e \to (M^e)^e$ and $N \to M^e \to (N^e)^e$, implies that M and N have the same exponent e^2, and so $e^2 = \pm 1$. But here our assumed exponent e is induced by a Galois automorphism, and then by Proposition 3 the only possibility is that n is even and $e = 2^{n/2}$. Under the isomorphism $M \to N^e$ for $e = n/2$ the trace $\eta + \eta^{-1}$ of xy is mapped onto the trace $\eta^2 + \eta^{-2}$ of $x'y^e$, so that $\eta^{n/2} + \eta^{-n/2} = \eta^2 + \eta^{-2}$. By our previous trace calculation the latter implies that $m \mid 2^{n/2} \pm 2$, so that m also divides $(2^{n/2} + 2)(2^{n/2} - 2) = 2^n - 4$. But at the same time $m \mid 2^n \pm 1$ and combining the two divisibility condition gives $m \mid 4 \pm 1$, contrary to our assumption that $m \geq 7$.

Thus, for distinct odd $k, m \geq 7$ such that $\gcd(k, m) \geq 7$ the parallel product $M \,||\, N$ is a non-orientable regular map of type $\{m, k\}$ with no exponents except ± 1. This completes the proof. □

6 Remarks

The theorems presented in this paper can also be seen as a demonstration of the usefulness of parallel products of maps in constructing new maps with given properties from suitable smaller ones. There are, however, limitations to this approach, and one has to be careful because some seemingly straightforward ideas might not work.

We illustrate this with reference to Theorem 2 of [19], which states that for every hyperbolic type $\{m, n\}$ there exists a reflexible orientably-regular map of type $\{m, n\}$, with exponent group $\{1, -1\}$. The proof uses residual finiteness of the triangle group, and hence has a non-constructive flavour. In an attempt to give a constructive proof using parallel products of maps, suppose instead that one takes an orientably-regular but chiral map M of type $\{m, n\}$ with trivial exponent group, and constructs the parallel product of M with its mirror image. In that case the resulting map is reflexible, but its exponent group can be larger than $\{-1, 1\}$.

For example, if M is the dual of the chiral map C46.6 from the list of chiral maps at [6], then M has type $\{25, 10\}$ and trivial exponent group, but the parallel product of M and its mirror image turns out to have exponent group $\{1, 3, 7, 9\}$, which is the entire group of units mod 10 (the valency of M).

Another limitation of our approach emerges by analysing the proof of Theorem 4. The key ingredient there was formation of a parallel product of the maps $M = \mathrm{Map}(\xi, \eta)$ and $N = \mathrm{Map}(\xi, \eta^2)$, and for this operation to make sense we needed the two maps to be non-isomorphic. Unfortunately, there are infinitely many counterexamples for hyperbolic pairs with coprime entries, even in a very strong sense that the Galois automorphism $z \mapsto z^2$ of $\mathrm{GF}(2^n)$ giving the power of 2 at η cannot be replaced by any other automorphism $z \mapsto z^{2^i}$ for $i < n$. More specifically, we will show that there are infinitely many pairs ξ, η of elements of suitable Galois fields of characteristic 2 with relatively prime odd orders $k = \mathrm{ord}(\xi)$ and $m = \mathrm{ord}(\eta)$ such that the maps $M = \mathrm{Map}(\xi, \eta)$ and $N = \mathrm{Map}(\xi, \eta^{2^i})$ are isomorphic for *every* integer i.

We begin with a technical observation. Keeping the notation of the proof of Theorem 4, for $G = \mathrm{SL}(2, q)$, $q = 2^n$, with ξ and η of distinct orders k and m, consider again the map $M = \mathrm{Map}(\xi, \eta) = (G; x, y)$. Further, for $i \in \{1 \dots, n-1\}$ let $M_i = \mathrm{Map}(\xi, \eta^{2^i}) = (G; x_i, y)$ be a regular map in which the generator x_i is obtained from x by replacing η with η^{2^i} in the definition (1) while letting ξ unchanged. Note that the same generator y appears in both M and M_i and that the maps have the same type $\{m, k\}$.

Observation 4.1. *The map M is isomorphic to M_i for some $i \in \{1, 2, \dots, n-1\}$ if and only if there exists an $\ell \in \{0, \dots, n-1\}$ such that $k \mid 2^\ell \pm 1$ and $m \mid 2^{\ell-i} \pm 1$, where $\ell - i$ is to be taken* mod n.

Proof. (Sketch.) Proceeding in a similar way as in Section 4, to establish an isomorphism from M onto M_i one has to find an element $\alpha \in \mathrm{SL}(2, q')$ for $q' \in \{q, q^2\}$ and a Galois automorphism $\theta :\ z \mapsto z^{2^\ell}$ of $\mathrm{GF}(q)$ for some $\ell \in \{0, \dots, n-1\}$ such that

$$\alpha x^\theta = x_i \alpha \quad \text{and} \quad \alpha y^\theta = y \alpha \ . \tag{6}$$

Leaving out details of the accompanying calculations, the second equation of (6) implies that either α is a diagonal matrix with diagonal entries a, a^{-1} for some non-zero $a \in \mathrm{GF}(q')$ and $\xi^{2^\ell} = \xi$ (which is equivalent to $k \mid 2^\ell - 1$), or α is an off-diagonal matrix with off-diagonal entries b, b^{-1} for some non-zero $b \in \mathrm{GF}(q')$ and $\xi^{2^\ell} = \xi^{-1}$ (which is equivalent to $k \mid 2^\ell + 1$).

In the diagonal case, from the first equation of (6) one further obtains $a = 1$ and then this equations holds if and only if $\eta^{2^\ell} + \eta^{-2^\ell} = \eta^{2^i} + \eta^{-2^i}$. By our previous trace calculations the latter is equivalent to $m = \mathrm{ord}(\eta) \mid 2^\ell \pm 2^i$, and interpreting powers of 2 as being taken mod n this condition can be written in the reduced form $m \mid 2^{\ell-i} \pm 1$ as m is odd. It follows that in the diagonal case existence of α and θ in (6) is equivalent to the arithmetic conditions $k \mid 2^\ell - 1$ and $m \mid 2^{\ell-i} \pm 1$.

In the off-diagonal case, the first equation of (6) implies that $b = \xi + \xi^{-1} + \eta^{2^\ell} + \eta^{-2^\ell}$, which is a non-zero value since $\xi + \xi^{-1}$ and $\eta^{2^\ell} + \eta^{-2^\ell}$ are traces of the elements x and $(xy)^{2^\ell}$ of different (and here even coprime) orders k and m. Also, $b \neq 0$ then implies the same condition $\eta^{2^\ell} + \eta^{-2^\ell} = \eta^{2^i} + \eta^{-2^i}$ as in the diagonal case, which simplifies the expression for b to $b = (\xi + \xi^{-1} + \eta + \eta^{-1})^{2^\ell}$. The off-diagonal case is then equivalent to $b = (\xi + \xi^{-1} + \eta + \eta^{-1})^{2^\ell}$ together with arithmetic conditions $k \mid 2^\ell + 1$ and $m \mid 2^{\ell-i} \pm 1$.

Summing up, the equations (6) are equivalent to the divisibility conditions $k \mid 2^\ell \pm 1$ and $m \mid 2^{\ell-i} \pm 1$, and the proof follows. $\square$

Corollary 2. *There are infinitely many pairs ξ, η of elements of suitable Galois fields of characteristic 2 with relatively prime odd orders $k = \mathrm{ord}(\xi)$ and $m = \mathrm{ord}(\eta)$ such that the maps $M = \mathrm{Map}(\xi, \eta)$ and $N = \mathrm{Map}(\xi, \eta^{2^i})$ are isomorphic for* every *integer i.*

Proof. Let $\kappa \geq 3$ and $\mu \geq 3$ be relatively prime positive integers and let $k = 2^\kappa - 1$ and $m = 2^\mu - 1$; by elementary number theory, $\gcd(\kappa, \mu) = 1$ implies $\gcd(k, m) = 1$. Clearly, κ and μ are the least integers with $k \mid 2^\kappa - 1$ and $m \mid 2^\mu - 1$, and relative primality of κ and μ implies that $n = \kappa\mu$ is the smallest positive integer such that both k and m divide $2^n \pm 1$ (note that neither k nor m divides any integer of the form $2^\nu + 1$ with $\nu < n$).

The parameter ℓ from Observation 4.1 is necessary a multiple of κ, so that $\ell = \kappa t$ for some $t \in \{1, 2, \ldots, \mu - 1\}$, and for such ℓ the divisibility condition $k \mid 2^\ell \pm 1$ from the observation is satisfied automatically (with a 'minus' sign). The second condition from Observation 4.1 reads $m \mid 2^{\ell - i} \pm 1$; since $m = 2^\mu - 1$, invoking number theory again one concludes that the condition is in this case equivalent to $\mu \mid \ell - i = \kappa t - i$.

We now show that for every $i \in \{1, 2, \ldots, \mu - 1\}$ the condition $\mu \mid \kappa t - i$ is satisfied for some $t \in \{1, 2, \ldots, \mu - 1\}$, and hence for some $\ell < n$. Indeed, by relative primality of κ and μ, for every $i \in \{1, 2, \ldots, \mu - 1\}$ there exist integers s and t such that $\mu s - \kappa t = i$; moreover, t can be chosen in such a way that $1 \leq t \leq \mu - 1$. By Observation 4.1 this proves our claim for maps $M = \mathrm{Map}(\xi, \eta)$ and $N = \mathrm{Map}(\xi, \eta^{2^i})$ for primitive k-th and m-th roots of unity ξ and η; note that the parameter i may be considered to be taken mod μ and for i a multiple of μ the statement is obvious. $\square$

We believe that the assumption of non-coprimality of entries of hyperbolic types in Theorem 4 may be removed, but this remains beyond methods developed in this paper.

Statements and Declarations

Data availability statement

This paper contains no data.

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All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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