GRAVIMETRIC QUASIGEOID IN SLOVAKIA BY THE FINITE ELEMENT METHOD

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The paper presents the solution to the geodetic boundary value problem by the finite element method in area of Slovak Republic. Generally, we have made two numerical experiments. In the first one, Neumann BC in the form of gravity disturbances generated from EGM-96 is used and the solution is verified by the quasigeoidal heights generated directly from EGM-96. In the second one, Neumann BC is computed from gravity measurements and the solution is compared to the quasigeoidal heights obtained by GPS/leveling method.

Keywords: finite element method, geodetic boundary value problem, ANSYS

AMS Subject Classification: 35Q80, 65N30, 86A30

1. INTRODUCTION

The Slovak Republic (SR), located between 47.6 and 49.6 deg north latitude and 16.3 and 22.5 deg longitude, is a country of about 49 000 km^2 with altitudes varying from 100 m to 2660 m above sea level. Such range of altitudes together with great variety of geological phenomena has led to the magnitudes of gravity anomalies between -25 and 130 mGal (Gal$\equiv 10^{-2}$ m$*$s^{-2}) and terrain corrections more than 35 mGal [10].

At the present, there exist several models of gravimetric quasigeoid in SR but neither one is ultimate. One approach uses classical methods – Fast Fourier Techniques for numerical integration of Stokes Formula (see [9, 10, 11]), while another explores the Boundary Element Method (BEM) [3, 4, 5, 6].

Our aim is also to compute the gravimetric quasigeoid precise enough to substitute the leveling method [17]. As a method of solution we have chosen the Finite Element Method (FEM). In this paper we present our first local solution of gravimetric quasigeoid in the area of the Slovak Republic.

2. DATA SOURCES

There were three types of data used in our computations, i. e., *global geopotential model* – EGM-96 [8], *terrestrial gravity data* – GrS-95 [7] based on 16 absolute gravity points and *data containing terrain information* – global – SRTM (Shuttle Radar

Topography Mission, http://srtm.usgs.gov/) and local – observation by NAVSTAR Global positioning system (GPS).

3. FORMULATION OF THE GEODETIC BOUNDARY VALUE PROBLEM

The quantity that describes the gravity field of the Earth is the *gravity potential* $W(\boldsymbol{x})$, consisting of the gravitational potential $V_g(\boldsymbol{x})$ generated by the Earth and the centrifugal potential $V_c(\boldsymbol{x})$ arisen from spinning of the Earth. In applications, it is also used an idealized (normal) model of the Earth (usually biaxial geocentric equipotential ellipsoid) rotating with the same angular velocity as the Earth. Its surface potential is equal to the potential on geoid (geoid is an equipotential surface which approximately coincides with the mean ocean surface) and its mass is the same as the mass of the Earth. Then the generated potential is called the *normal potential* and denoted $U(\boldsymbol{x})$. The difference between the actual gravity potential and this normal potential is called the *disturbing potential* $T(\boldsymbol{x})$ [12, 17]. Neglecting the atmosphere, the disturbing potential is a harmonic function outside the Earth, i. e., it satisfies the Laplace equation $\Delta T(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in R^3 - E$, where E represents the Earth.

The development of satellite technologies has brought new opportunities in geodesy. With gravimetric and satellite measurements we can get the magnitude $\gamma(\boldsymbol{x})$ of the normal gravity vector $\vec{\gamma}(\boldsymbol{x})$ as well as the magnitude $g(\boldsymbol{x})$ of the actual gravity $\vec{g}(\boldsymbol{x})$ at the same point. This way we get so-called the *gravity disturbance* $\delta g(\boldsymbol{x}) = g(\boldsymbol{x}) - \gamma(\boldsymbol{x}), \boldsymbol{x} \in R^3$.

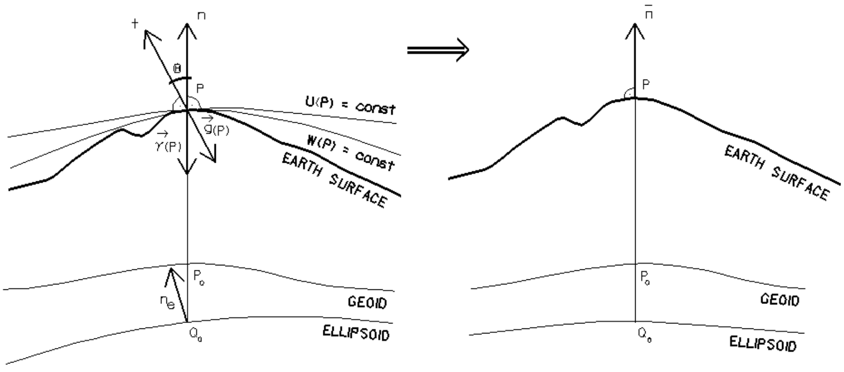

Fig. 1. Gravity and normal gravity vectors, deflection of vertical.

Applying the gradient operator to the definition of the disturbing potential we get $\nabla T(\boldsymbol{x}) = \nabla W(\boldsymbol{x}) - \nabla U(\boldsymbol{x}) = \vec{g}(\boldsymbol{x}) - \vec{\gamma}(\boldsymbol{x}), \boldsymbol{x} \in R^3$. In Figure 1, we plot the equipotential surfaces $U(P) = \text{const}$, $W(P) = \text{const}$ that pass through the same point P on the Earth surface. There we can observe differences in vectors $\vec{\gamma}(\boldsymbol{x})$ and $\vec{g}(\boldsymbol{x})$ that have the opposite direction to the normal n and plumb line t. The spatial angle θ, which is called the deflection of vertical, is negligibly small in real situation. In addition, directions of both vectors are very close to the opposite direction of the outer normal n_e to the ellipsoid. Neglecting the small angles (which are less than minute) we can project the gradient of the disturbing potential to the Earth normal $\bar{n}$ and get $\langle \nabla T(\boldsymbol{x}), \bar{n}\rangle = \langle \nabla W(\boldsymbol{x}), \bar{n}\rangle - \langle \nabla U(\boldsymbol{x}), \bar{n}\rangle$, which is approximately equal to $\langle \nabla W(\boldsymbol{x}), t\rangle - \langle \nabla U(\boldsymbol{x}), n\rangle = -g(\boldsymbol{x}) + \gamma(\boldsymbol{x}) = -\delta g(\boldsymbol{x})$. This idea has been used by Čunderlík, Mikula, Mojzeš in [3, 4, 5, 6], where the Neumann Boundary Value Problem (BVP) on infinite domain with a boundary condition given on the Earth's surface in the form of gravity disturbance were suggested and solved numerically by the BEM.

Although the given problem deals with the infinite domain in our approach using the FEM, we construct an artificial boundary away from the approximate Earth surface, see Figure 2, and due to giant size of the Earth we restrict our computa-

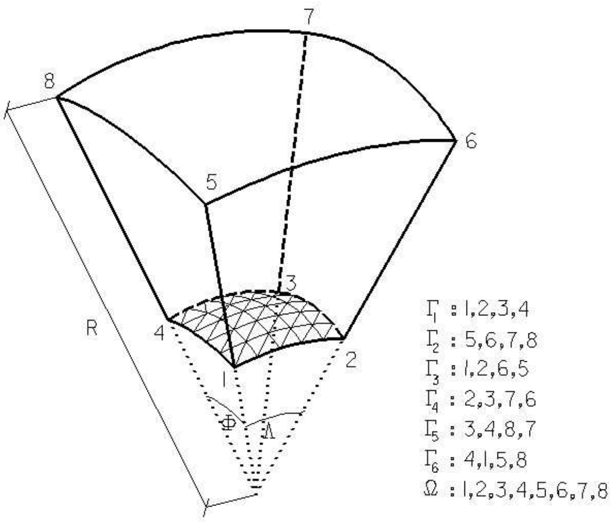

Fig. 2. Geometry of computational domain.

tions only to a partial domain Ω depicted in Figure 2, too. The bottom surface Γ_1 represents a part of the real Earth surface, discretized by triangles, covering e. g. a neighborhood of Slovakia, where the Neumann boundary condition is given. The upper spherical part $\Gamma_2 = \{x; |x| = R\}$ of the domain represents the artificial boundary where the Dirichlet boundary condition is prescribed and also on further artificial planar boundaries $\Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$ we use Dirichlet boundary condition. Then our geodetic BVP is defined as follows:

$$-\Delta T(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega, \tag{1}$$

$$\frac{\partial T}{\partial \bar{n}} = -\delta g(\boldsymbol{x}) \quad \text{on} \quad \Gamma_1, \tag{2}$$

$$T(\boldsymbol{x}) = T_{\mathrm{EGM-96}}(\boldsymbol{x}), \quad \text{on} \quad \Gamma_i, \quad i = 2, \ldots, 6, \tag{3}$$

where $T_{\mathrm{EGM-96}}$ represents the disturbing potential generated from global geopotential model EGM-96.

4. SOLUTION OF THE GEODETIC BOUNDARY VALUE PROBLEM

4.1. Weak formulation of the problem

In order to construct the weak formulation of the problem (1) – (3) we define the space of test functions V that is the space of functions from $W_2^{(1)}(\Omega)$ which are equal to 0 on Γ_i, $i = 2, \ldots, 6$ in the sense of traces. We multiply the differential equation (1) by $w \in V$ and get

$$\int_\Omega \nabla T \cdot \nabla w \,\mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{\partial\Omega} \nabla T \cdot nw \,\mathrm{d}\sigma = 0, \quad \forall\, w \in V. \tag{4}$$

We can define the weak formulation of our BVP (1) – (3) as follows: we look for a function T, such that $T - T_{\mathrm{EGM-96}} \in V$ and

$$\int_\Omega \nabla T \cdot \nabla w \,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \int_{\Gamma_1} \delta g w \,\mathrm{d}\sigma = 0, \quad \forall\, w \in V. \tag{5}$$

Assuming $\delta g \in L^2(\Gamma_1)$ and extension of $T_{\mathrm{EGM-96}} \in W_2^{(1)}(\Omega)$ due to [2] or [15] solution of this problem always exists and is unique.

Moreover the finite element approximation given in the following subsection converges to the weak solution refining the finite element grid.

4.2. FEM discretization

The FEM is a numerical method for solving partial differential equations and it assumes discretization of the domain by a set of subdomains called the finite elements see e. g. [2] or [14]. In order to build the finite element approximation one can proceed as follows. One seeks an approximation to the solution of the differential equation on every element e as a linear combination of nodal values t_j^e and approximation functions Ψ_j^e i. e.,

$$T(x,y,z) \approx T^e(x,y,z) = \sum_{j=1}^{N} t_j^e \Psi_j^e(x,y,z), \tag{6}$$

where N is a number of element nodes. Then the weak identity (4) is used on every element and $\Psi_i^e, i = 1, \ldots, N$ are considered as test functions w. In this way we get the element system of equations for each e given by

$$\int_{\Omega^e} \sum_{j=1}^{N} t_j \nabla\Psi_i^e \cdot \nabla\Psi_j^e \mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{\partial\Omega^e} \Psi_i^e q_n \mathrm{d}\sigma = 0, \quad i = 1, \ldots, N, \tag{7}$$

where q_n are interelement fluxes. It can be written in a matrix form: $\sum_{j=1}^{N} K_{ij}^e t_j - Q_i^e = 0$, where $K = [K_{ij}^e]_{N \times N}$ is an element stiffness matrix and $Q = [Q_i^e]_N$ represents a vector of fluxes through element faces.

To create the global finite element model, one should use the balance of the interelement fluxes and continuity of numerical solution on interelement boundaries. Taking into account the boundary conditions (2) – (3) one ends up with the global linear system. For our 3D case we use finite element software ANSYS [1] and its 3D 4-nodes tetrahedral elements.

5. NUMERICAL EXPERIMENTS

Our computational domain Ω has been the space above Slovakia, represented by a series of triangular areas with maximal diameter 0.11 deg, up to the sphere with radius $R = 6500$ or 7000 km. On one spherical and four planar boundaries of Ω we consider the Dirichlet BC, i. e., disturbing potential generated from EGM-96 [8] and on the bottom boundary the Neumann BC in the form of gravity disturbances is used.

In the first experiment the input gravity disturbances (2), shown in the second column of Figure 3, were generated from EGM-96 by program GRAFIM [13]. The ANSYS nodal solution, the disturbing potential T, has been transformed into the quasigeoidal heights by the Bruns formula [12] and compared with quasigeoidal heights generated from EGM-96 using program GRAFIM as well. One can see good qualitative agreement of our result in comparison to heights generated from EGM-96 (Figure 4).

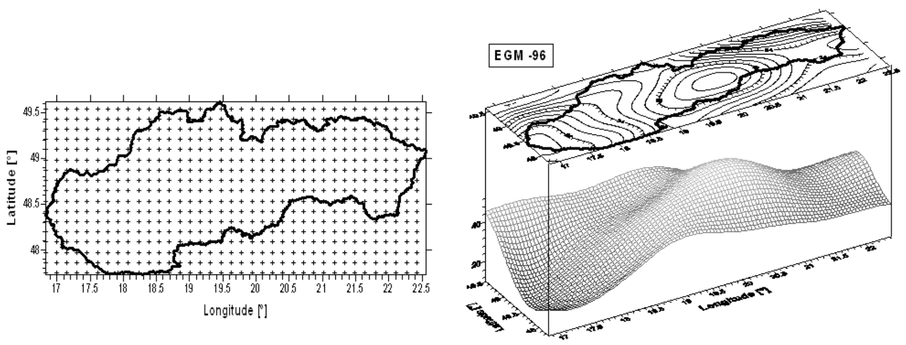

Fig. 3. Computational grid and gravity disturbances generated from EGM-96.

In the second experiment we have used the gravity disturbances (2) computed from actual gravity [7], depicted in Figure 5. As it is obvious from Figure 3 and Figure 5, there were differences between these two types of data and as well as between the following final quasigeoidal heights (Figure 6). The reason is that the EGM-96, from which the gravity disturbances in the first experiment were generated, is joint spherical harmonic model completed to the degree and order 360 and on the other hand the actual gravities used in the second experiment were gained by detailed gravity mapping.

The results of the second experiment have been tested by 61 GPS/leveled points

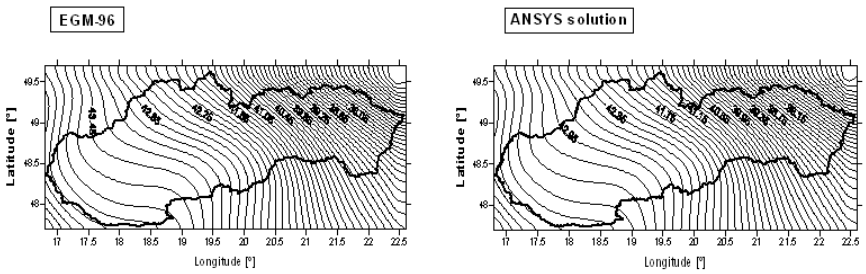

Fig. 4. Quasigeoidal heights generated from EGM-96 and computed by ANSYS from EGM-96 gravity disturbances.

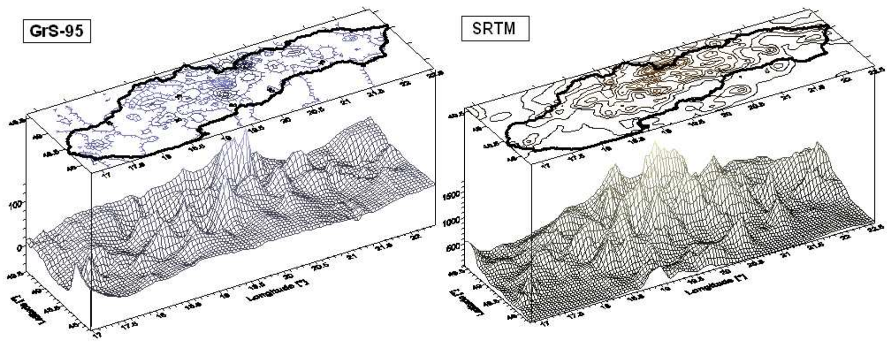

Fig. 5. Gravity disturbances computed from detailed gravity mapping GrS-95 and model of terrain generated from SRTM.

shown in Figure 7. To gain ellipsoidal heights H the 36-hours GPS observations' sessions were performed and for altitudes above sea level h all points were connected to the Slovak leveling network using precise spirit leveling. Then for reference quasigeoidal heights hold $\zeta_{\mathrm{ref}} = h - H$.

The mean residual $\zeta_{\mathrm{ref}} - \zeta_{\mathrm{ANSYS}}$ has been $0.803\,\mathrm{m}$ and the standard deviation $0.202\,\mathrm{m}$. It is a promising result in comparison with two versions of much more detailed Gravimetric Model of Slovak Quasigeoid 2003 (GMSQ03B/C) where the mean residuals have been $0.334\,\mathrm{m}$ (GMSQ03B) and $0.711\,\mathrm{m}$ (GMSQ03C) and the standard deviations $0.190\,\mathrm{m}$ (GMSQ03B) and $0.076\,\mathrm{m}$ (GMSQ03C) [9].

In order to use the quasigeoid model in practice the adaptation into the national vertical datum, so called fitting, has to be made. Our future investigation will be aimed at both fitting and improving of accuracy of the quasigeoid model by using genuine theory in detailed terrain and gravimetric data. We are willing to use parallel computations as well.

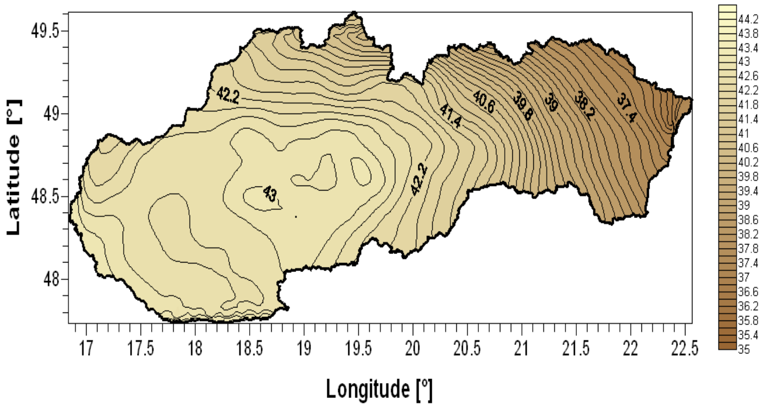

Fig. 6. Gravimetric model of Slovak quasigeoid calculated by FEM.

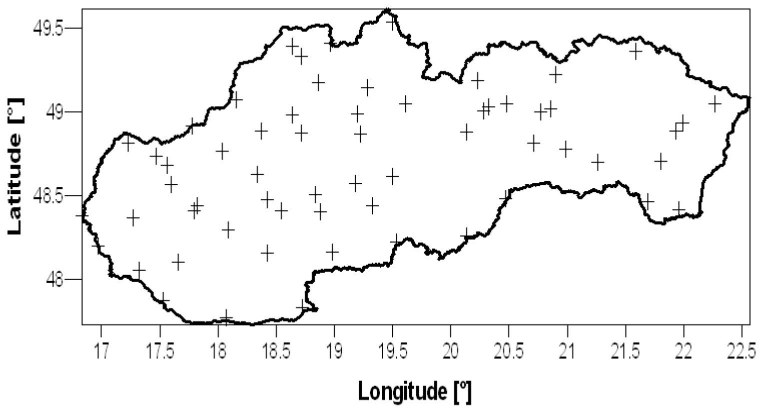

Fig. 7. Distribution of the testing points.

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