Aggregation procedure and fuzzy relation properties based on binary operations

Urszula Bentkowska and Anna Król

University of Rzeszów
Poland

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Contents

Preliminaries

Definition 1 (cf. T. Calvo et al.[1]). Let $n \in \mathbb{N}$. A function $A : [0,1]^n \to [0,1]$ which is increasing, i.e.

$$A(x_1, \dots, x_n) \leqslant A(y_1, \dots, y_n) \quad \text{for} \quad x_i, y_i \in [0,1], x_i \leqslant y_i, i = 1, \dots, n$$

is called an aggregation function if

$$A(0, \dots, 0) = 0,\ A(1, \dots, 1) = 1.$$

Example 1. Let $t_1, ..., t_n, w_1, ..., w_n \in [0,1]$. Aggregation functions are:

- the weighted minimum

$$F(t_1, ..., t_n) = \min_{1 \leqslant k \leqslant n} \max(1 - w_k, t_k),\ \max_{1 \leqslant k \leqslant n} w_k = 1,$$

- the weighted maximum

$$F(t_1, ..., t_n) = \max_{1 \leqslant k \leqslant n} \min(w_k, t_k),\ \max_{1 \leqslant k \leqslant n} w_k = 1,$$

[1] T. Calvo, A. Kolesárová, M. Komorníková and R. Mesiar, Aggregation operators: properties, classes and construction methods, In T. Calvo, G. Mayor and R. Mesiar, eds., *Aggregation Operators vol. 97: Studies in Fuzziness and Soft Computing*, pp. 3–104, Physica-Verlag, Heildelberg, 2002

- the median value

$$\operatorname{med}(t_1, \ldots, t_n) = \begin{cases} \frac{s_k + s_{k+1}}{2}, & \text{for } n = 2k \\ s_{k+1}, & \text{for } n = 2k+1 \end{cases},$$

where $(s_1, \ldots, s_n)$ is the increasingly ordered sequence of the values $t_1, \ldots, t_n$, i.e. $s_1 \leqslant \ldots \leqslant s_n$.

Other examples of aggregation functions are:

- geometric mean

$$G(x_1, \ldots, x_n) = \sqrt[n]{x_1 \cdot \ldots \cdot x_n},$$

- weighted means

$$A_w(x_1, \ldots, x_n) = \sum_{k=1}^{n} w_k x_k, \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1,$$

- quasi-arithmetic means

$$M_\varphi(x_1, \ldots, x_n) = \varphi^{-1}(\frac{1}{n} \sum_{k=1}^{n} \varphi(x_k)),$$

• quasi-linear means

$$F(x_1, ..., x_n) = \varphi^{-1}(\sum_{k=1}^{n} w_k \varphi(x_k)), \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1,$$

where $x_1, \ldots, x_n \in [0, 1]$, $\varphi : [0, 1] \to \mathbb{R}$ is a continuous, strictly increasing function.

Definition 2. Let $n \in \mathbb{N}$. We say that a function $F : [0, 1]^n \to [0, 1]$:
• has a zero element $z \in [0, 1]$ if for each $k \in \{1, \ldots, n\}$ and each $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n \in [0, 1]$ one has

$$F(x_1, ..., x_{k-1}, z, x_{k+1}, ..., x_n) = z,$$

• is without zero divisors if

$$\forall_{x_1,...,x_n \in [0,1]} \quad (F(x_1, ..., x_n) = z \ \Rightarrow \ (\exists_{1 \leqslant k \leqslant n} \ x_k = z)).$$

Definition 3 (Drewniak, Król 2010[2]). An operation $C : [0, 1]^2 \to [0, 1]$ is called a fuzzy conjunction if it is increasing with respect to each variable and

$$C(1, 1) = 1, \quad C(0, 0) = C(0, 1) = C(1, 0) = 0.$$

[2] J. Drewniak, A. Król, A survey of weak connectives and the preservation of their properties by aggregations, Fuzzy Sets and Systems, 161 (2010), 202–215.

An operation $D : [0,1]^2 \to [0,1]$ is called a fuzzy disjunction if it is increasing with respect to each variable and

$$D(0,0) = 0, \quad D(1,1) = D(0,1) = D(1,0) = 1.$$

Corollary 1. *A fuzzy conjunction has a zero element* 0*. A fuzzy disjunction has a zero element* 1*.*

Example 2. Consider the following family of fuzzy conjunctions for $\alpha \in [0,1]$

$$C^\alpha(x,y) = \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \\ \alpha & \text{otherwise} \end{cases}.$$

Operations C^0 and C^1 are the least and the greatest fuzzy conjunction, respectively.
A fuzzy conjunction C^α has no zero divisors if and only if $\alpha \in (0,1]$.
Other examples of fuzzy conjunction without zero divisors are:

$$C(x,y) = \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \\ x & \text{otherwise} \end{cases}, \qquad C(x,y) = \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \\ y & \text{otherwise} \end{cases}.$$

Definition 4 (E. P. Klement et al.[3]). A triangular norm $T : [0,1]^2 \to [0,1]$ (triangular conorm $S : [0,1]^2 \to [0,1]$) is an arbitrary associative, commutative, increasing in each variable operation having a neutral element $e = 1$ ($e = 0$).

Corollary 2. *A triangular norm (conorm) has a zero element* $z = 0$ *(*$z = 1$*).*

Example 3 (E. P. Klement et al.[3]). The four well-known examples of t-norms T and corresponding t-conorms S are:

$$T_M(s,t) = \min(s,t), \qquad T_P(s,t) = st,$$
$$S_M(s,t) = \max(s,t), \qquad S_P(s,t) = s + t - st,$$

$$T_L(s,t) = \max(s+t-1, 0), \qquad S_L(s,t) = \min(s+t, 1),$$

$$T_D(s,t) = \begin{cases} s, & t = 1 \\ t, & s = 1 \\ 0, & otherwise \end{cases}, \quad S_D(s,t) = \begin{cases} s, & t = 0 \\ t, & s = 0 \\ 1, & otherwise \end{cases}$$

for $s, t \in [0,1]$.

[3] E.P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Kluwer Acad. Publ., Dordrecht, 2000

Definition 5 (E. P. Klement et al.[3]). A strict t-norm $T : [0,1]^2 \to [0,1]$ is a t-norm which is continuous and strictly increasing in $(0,1]^2$.

Theorem 1 (J. Fodor, M. Roubens[4]). *Any strict t-norm T is a function isomorphic to the product t-norm T_P, i.e.*

$$T(x_1, x_2) = \varphi^{-1}(T_P(\varphi(x_1), \varphi(x_2))), \quad x_1, x_2 \in [0,1],$$

where $\varphi : [0,1] \to [0,1]$ is an increasing bijection.

Corollary 3. *Triangular norms:* min, T_P, *strict t-norms are functions without zero divisors.*

Definition 6 (T. Calvo et al.[5]). Let $F : [0,1]^n \to [0,1]$. A function F^d is called a dual function to F, if for all $x_1, \ldots, x_n \in [0,1]$

$$F^d(x_1, \ldots, x_n) = 1 - F(1 - x_1, \ldots, 1 - x_n).$$

F is called a self-dual function, if it holds $F = F^d$.

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[4] J. Fodor, M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer Acad. Publ., Dordrecht, 1994

[5] T. Calvo, A. Kolesárová, M. Komorníková and R. Mesiar, Aggregation operators: properties, classes and construction methods, In T. Calvo, G. Mayor and R. Mesiar, eds., *Aggregation Operators vol. 97: Studies in Fuzziness and Soft Computing*, pp. 3–104, Physica-Verlag, Heildelberg, 2002

Domination

Definition 7 (cf. B. Schweizer, A. Sklar[6]). Let $m, n \in \mathbb{N}$. A function $F\colon [0,1]^m \to [0,1]$ dominates function $G\colon [0,1]^n \to [0,1]$ ($F \gg G$) if for an arbitrary matrix $[a_{ik}] = A \in [0,1]^{m\times n}$ the following inequality holds

$$F(G(a_{11},...,a_{1n}),...,G(a_{m1},...,a_{mn})) \geqslant G(F(a_{11},...,a_{m1}),...,F(a_{1n},...,a_{mn})).$$

Theorem 2 (cf. S. Saminger et al.[7]). *A function $F\colon [0,1]^n \to [0,1]$, which is increasing in each of its arguments dominates minimum if and only if for each $t_1,...,t_n \in [0,1]$*

$$F(t_1,...,t_n) = \min(f_1(t_1),...,f_n(t_n)),$$

where $f_k\colon [0,1] \to [0,1]$ is increasing with $k = 1,...,n$.

[6] B. Schweizer, A. Sklar, Probabilistic metric spaces, North Holland, New York, 1983.

[7] S. Saminger, R. Mesiar and U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, *Internat. J. Uncertain., Fuzziness, Knowl.-Based Syst.*, 10(Suppl.):11–35, World Scientific, 2002

Example 4 (cf. J. Drewniak, U. Dudziak[8], cf. S. Saminger et al.[9]). The weighted geometric mean dominates t-norm T_P. The weighted arithmetic mean dominates t-norm T_L. The function

$$F(t_1, ..., t_n) = \frac{p}{n} \sum_{k=1}^{n} t_k + (1-p) \min_{1\leqslant k\leqslant n} t_k \tag{1}$$

dominates T_L, where $p \in (0,1)$. The weighted minimum dominates every t-norm T. Let us consider projections P_k. Then $F \gg P_k$ and $P_k \gg F$ for any function $F : [0,1]^n \to [0,1]$. Minimum dominates any fuzzy conjunction.

[8] J. Drewniak, U. Dudziak, Preservation of properties of fuzzy relations during aggregation processes, *Kybernetika*, 43(2):115–132, Institute of Information Theory and Automation Academy of Sciences of Czech Republic, 2007

[9] S. Saminger, R. Mesiar and U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, *Internat. J. Uncertain., Fuzziness, Knowl.-Based Syst.*, 10(Suppl.):11–35, World Scientific, 2002

Lemma 1 (J. Drewniak, A. Król[10]). *Let* $F : [0,1]^n \to [0,1]$, $F : [0,1]^n \to [0,1]$. *If* $F \gg G$, *then* $G^d \gg F^d$.

Corollary 4. *Let* $F : [0,1]^n \to [0,1]$, T *be a t-norm,* S *be a corresponding dual t-conorm,* C *be a fuzzy conjunction and* D *be a corresponding dual fuzzy disjunction. If* $F \gg T$, *then* $S \gg F^d$. *If* $F \gg C$, *then* $D \gg F^d$.

Example 5. The weighted arithmetic mean are dominated by S_L and the weighted maximum is dominated by any t-conorm S. Moreover, the function

$$F(t_1, ..., t_n) = \frac{p}{n} \sum_{k=1}^{n} t_k + (1-p) \max_{1 \leqslant k \leqslant n} t_k \tag{2}$$

is dominated by S_L, where $p \in (0,1)$.

[10] J. Drewniak and A. Król, On the problem of domination between triangular norms and conorms, *Journal of Electrical Engineering*, 56(12/s):59–61, Slovak Centre of IEE, 2005

A function $F\colon [0,1]^n \to [0,1]$, which is increasing in each of its arguments is dominated by maximum if and only if for each $t_1, ..., t_n \in [0,1]$

$$F(t_1, ..., t_n) = \max(f_1(t_1), ..., f_n(t_n)),$$

where $f_k\colon [0,1] \to [0,1]$ is increasing with $k = 1, ..., n$. Examples of such functions are:
if $f_k(t) = t$, $k = 1, ..., n$, then $F = \max$,
if for a certain $k \in \{1, ..., n\}$, function $f_k(t) = t$ and $f_i(t) = 1$ for $i \neq k$, then $F = P_k$ - projections, if $f_k(t) = \min(v_k, t)$, $v_k \in [0,1]$, $k = 1, ..., n$, $\max\limits_{1 \leqslant k \leqslant n} v_k = 1$, then F is the weighted maximum.

Fuzzy relations

Definition 8 (L.A. Zadeh[11]). A fuzzy relation on a set $X \neq \emptyset$ is an arbitrary function $R : X \times X \to [0,1]$. The family of all fuzzy relations on X is denoted by $FR(X)$.

Definition 9 (cf. J. Fodor, M. Roubens[12]). Let $B, B_1, B_2 : [0,1]^2 \to [0,1]$ be binary operations. Relation $R \in FR(X)$ is:

- B-asymmetric, if $\underset{x,y \in X}{\forall} B(R(x,y), R(y,x)) = 0$,
- B-antisymmetric, if
$\underset{x,y,x \neq y \in X}{\forall} B(R(x,y), R(y,x)) = 0$,
- totally B-connected, if
$\underset{x,y \in X}{\forall} B(R(x,y), R(y,x)) = 1$,
- B-connected, if $\underset{x,y,x \neq y \in X}{\forall} B(R(x,y), R(y,x)) = 1$,
- B-transitive, if
$\underset{x,y,z \in X}{\forall} B(R(x,y), R(y,z)) \leqslant R(x,z)$,

[11] L.A. Zadeh, Fuzzy sets, *Inform. Control*, 8:338–353, Elsevier, 1965

[12] J. Fodor, M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer Acad. Publ., Dordrecht, 1994

- negatively B-transitive, if
$$\forall_{x,y,z \in X} \; B(R(x,y), R(y,z)) \geqslant R(x,z),$$
- B_1-B_2-Ferrers, if
$$\forall_{x,y,z,w \in X} \; B_1(R(x,y), R(z,w)) \leqslant B_2(R(x,w), R(z,y)),$$
- B_1-B_2-semitransitive, if
$$\forall_{x,y,z,w \in X} \; B_1(R(x,w), R(w,y)) \leqslant B_2(R(x,z), R(z,y)).$$

Definition 10 (S. Saminger et al.[13]**).** Let $F : [0,1]^n \to [0,1]$, $R_1, \ldots, R_n \in FR(X)$. An aggregated fuzzy relation $R_F \in FR(X)$ is described by the formula

$$R_F(x,y) = F(R_1(x,y), \ldots, R_n(x,y)), \quad x, y \in X.$$

A function F preserves a property of fuzzy relations if for every $R_1, \ldots, R_n \in FR(X)$ having this property, R_F also has this property.

[13] S. Saminger, R. Mesiar and U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, *Internat. J. Uncertain., Fuzziness, Knowl.-Based Syst.*, 10(Suppl.):11–35, World Scientific, 2002

Theorem 3 (cf. J. Drewniak, U. Dudziak[14]). *Let card $X \geqslant 2$, B have a zero element 0 and be without zero divisors. A function F preserves B-asymmetry (B-antisymmetry) if and only if it satisfies the following condition for all $s, t \in [0,1]^n$*

$$\underset{1 \leqslant k \leqslant n}{\forall} \min(s_k, t_k) = 0 \Rightarrow \min(F(s), F(t)) = 0. \tag{3}$$

Example 6 (cf. J. Drewniak, U. Dudziak[14]). Let B be a fuzzy conjunction without zero divisors (e.g. a strict t-norm). The function $F = \min$ preserves B-asymmetry (B-antisymmetry). Functions F which has the zero element $z = 0$ with respect to certain coordinate, i.e.

$$\underset{1 \leqslant k \leqslant n}{\exists} \ \underset{i \neq k}{\forall} \ \underset{t_i \in [0,1]}{\forall} F(t_1, \ldots, t_{k-1}, 0, t_{k+1}, \ldots, t_n) = 0$$

fulfil (3), so they preserve B-asymmetry (B-antisymmetry). In particular, the weighted geometric mean fulfil (3). As another example we may consider the median function. If a function F fulfils the following condition then we also get (3).

$$\underset{t \in [0,1]^n}{\forall} \ \mathrm{card}\{k : t_k = 0\} > \frac{n}{2} \Rightarrow F(t) = 0. \tag{4}$$

[14] J. Drewniak, U. Dudziak, Preservation of properties of fuzzy relations during aggregation processes, *Kybernetika*, 43(2):115–132, 2007

Theorem 4 (cf. J. Drewniak, U. Dudziak[15]). *Let card* $X \geqslant 2$, B *have a zero element* 1 *and be without zero divisors. A function* F *preserves total* B*-connectedness (*B*-connectedness) if and only if it satisfies the following condition for all* $s, t \in [0,1]^n$

$$\underset{1\leqslant k\leqslant n}{\forall} \max(s_k, t_k) = 1 \Rightarrow \max(F(s), F(t)) = 1. \tag{5}$$

Example 7 (cf. J. Drewniak, U. Dudziak[15]). Let B be a fuzzy disjunction without zero divisors (e.g. a strict t-conorm). Examples of functions fulfilling (5) for all $s, t \in [0,1]^n$ are $F = \max$, $F =$med or functions F with the zero element $z = 1$ with respect to a certain coordinate, i.e.

$$\underset{1\leqslant k\leqslant n}{\exists} \ \underset{i\neq k}{\forall} \ \underset{t_i\in[0,1]}{\forall} F(t_1, \ldots, t_{k-1}, 1, t_{k+1}, \ldots, t_n) = 1.$$

The dual property for (4) have the form

$$\underset{t\in[0,1]^n}{\forall} \ \mathrm{card}\{k : t_k = 1\} > \frac{n}{2} \Rightarrow F(t) = 1.$$

[15] J. Drewniak, U. Dudziak, Preservation of properties of fuzzy relations during aggregation processes, *Kybernetika*, 43(2):115–132, 2007

Theorem 5 (cf. J. Drewniak, U. Dudziak[16]). *Let card* $X \geqslant 3$, *B has a zero element* $z = 0$. *If a function* $F\colon [0,1]^n \to [0,1]$ *preserves* B*-transitivity, then it dominates* B *(*$F \gg B$*), it means that for all* $(s_1, ..., s_n), (t_1, ..., t_n) \in [0,1]^n$

$$F(B(s_1, t_1), \ldots, B(s_n, t_n)) \geqslant B(F(s_1, \ldots, s_n), F(t_1, \ldots, t_n)).$$

Theorem 6 (cf. J. Drewniak, U. Dudziak[16]). *If a function* $F\colon [0,1]^n \to [0,1]$, *which is increasing in each of its arguments fulfils* $F \gg B$, *then it preserves* B*-transitivity.*

Example 8 (cf. S. Saminger et al.[17]). Minimum dominates any fuzzy conjunction C. Each quasi–linear mean dominates T_D. Moreover, for $n = 2$ arbitrary t-norm $F = T$ dominates T_D. The minimum, the weighted minimum and the projections dominate min. The weighted geometric mean preserves T_P-transitivity, the weighted arithmetic mean preserves T_L-transitivity, the minimum preserves T-transitivity with arbitrary t-norm T. The function F described by the formula (1) preserves T_L-transitivity.

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[16] J. Drewniak, U. Dudziak, Preservation of properties of fuzzy relations during aggregation processes, *Kybernetika*, 43(2):115–132, 2007

[17] S. Saminger, R. Mesiar and U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, *Internat. J. Uncertain., Fuzziness, Knowl.-Based Syst.*, 10(Suppl.):11–35, 2002

Theorem 7. *Let card* $X \geqslant 3$, B *has a zero element* $z = 1$. *An increasing in each of its arguments function* $F\colon [0,1]^n \to [0,1]$ *preserves negative* B*-transitivity if and only if* $B \gg F$, *it means that for any* $(s_1, ..., s_n)$, $(t_1, ..., t_n) \in [0,1]^n$ *the following inequality holds*

$$B(F(s_1, \ldots, s_n), F(t_1, \ldots, t_n)) \geqslant F(B(s_1, t_1), \ldots, B(s_n, t_n)). \tag{6}$$

Example 9. The maximum preserves negative D-transitivity for any disjunction D. The weighted maximum preserves negative S-transitivity for any t-conorm S. The weighted arithmetic mean and the functions of the form (2) preserve negative S_L-transitivity. Moreover, negative S_M-transitivity (i.e. negative transitivity) is preserved if and only if a function F is of the form

$$F(t_1, ..., t_n) = \max(f_1(t_1), ..., f_n(t_n)),$$

for each $t_1, ..., t_n \in [0,1]$ and $f_k\colon [0,1] \to [0,1]$ being increasing with $k = 1, ..., n$.

Theorem 8. *If a function* $F\colon [0,1]^n \to [0,1]$*, which is increasing in each of its arguments fulfils* $F \gg B_1$ *and* $B_2 \gg F$*, then it preserves* B_1*-*B_2*-Ferrers property.*

Theorem 9. *If a function* $F\colon [0,1]^n \to [0,1]$*, which is increasing in each of its arguments fulfils* $F \gg B_1$ *and* $B_2 \gg F$*, then it preserves* B_1*-*B_2*-semitransitivity.*

Example 10. The weighted arithmetic mean preserve B_1-B_2-Ferrers property and B_1-B_2-semitransitivity for t-norm $T_L = B_1$ and t-conorm $S_L = B_2$.

Example 11. Conditions given in Theorems 8 and 9 are only the sufficient ones. Let us consider function $F(s,t) = st$ (so $F = T_P$) and fuzzy relations presented by the matrices

$$R_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Relations R_1, R_2 are min-max-Ferrers (J. Fodor, M. Roubens[18]) and min-max-semitransitive. Moreover $R = F(R_1, R_2)$ is both min-max-Ferrers and min-max-semitransitive, where $R \equiv 0$. However, it is not true that $F \gg \min$ (the only t-norm that dominates minimum is minimum itself).

[18] J. Fodor, M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer Acad. Publ., Dordrecht, 1994